MODERN
MATHEMATICAL
TEXTS

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ELEMENTARY MATHEMATICAL ANALYSIS

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EDITED BY CHARLES S. SLICHTER

ELEMENTARY MATHEMATICAL ANALYSIS

A TEXT BOOK FOR FIRST YEAR COLLEGE STUDENTS

BY

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PREFACE

THIS book is not intended to be a text on "Practical Mathematics" in the sense of making use of scientific material and of fundamental notions not already in the possession of the student, or in the sense of making the principles of mathematics secondary to its technique. On the contrary, it has been the aim to give the fundamental truths of elementary analysis as much prominence as seems possible in a working course for freshmen.

The emphasis of the book is intended to be upon the notion of *functionality*. Illustrations from science are freely used to make this concept prominent. The student should learn early in his course that an important purpose of mathematics is to express and to interpret the laws of actual phenomena and not primarily to secure here and there certain computed results. Mathematics might well be defined as the science that takes the broadest view of all of the sciences—an epitome of quantitative knowledge. The introduction of the student to a broad view of mathematics can hardly begin too early.

The ideas explained above are developed in accordance with a two-fold plan, as follows:

First, the plan is to group the material of elementary analysis about the consideration of the three fundamental functions:

1. The Power Function $y = ax^n$ (n any number) or the law "*as x changes by a fixed multiple, y changes by a fixed multiple also.*"

2. The Simple Periodic Function $y = a \sin mx$, considered as fundamental to all periodic phenomena.

3. The Exponential Function, or the law "*as x changes by a fixed increment, y changes by a fixed multiple.*"

Second, the plan is to enlarge the elementary functions by the development of the fundamental *transformations* applicable to these and other functions. To avoid the appearance of abstruseness, these transformations are stated with respect to the *graphs* of the functions; that is, they are not called *transformations*, but "motions" of the loci. The facts are summarized in several simple "Theorems on Loci," which explain the translation, rotation, shear, and elongation or contraction of the graph of any function in the xy plane.

Combinations of the fundamental functions as they actually occur in the expression of elementary natural laws are also discussed and examples are given of a type that should help to explain their usefulness.

Emphasis is placed upon the use of *time* as a variable. This enriches the treatment of the elementary functions and brings many of the facts of "analytic geometry" into close relation to their application in science. A chapter on waves is intended to give the student a broad view of the use of the trigonometric functions and an introduction to the application of analysis to periodic phenomena.

It is difficult to understand why it is customary to introduce the trigonometric functions to students seventeen or eighteen years of age by means of the restricted definitions applicable only to the right triangle. Actual test shows that such rudimentary methods are wasteful of time and actually confirm the student in narrowness of view and in lack of scientific imagination. For that reason, the definitions, theorems and addition formulas of trigonometry are kept as general as practicable and the formulas are given general demonstrations.

The possibilities and responsibilities of character building in the department of mathematics are kept constantly in mind. It is accepted as fundamental that a modern working course in mathematics should emphasize proper habits of work as well as proper methods of thought; that neatness, system, and orderly habits have a high value to all students of the sciences, and that a textbook should help the teacher in every known way to develop these in the student.

Chapters V, VI and VII contain material that is required for admission to many colleges and universities. The amount of time devoted to these chapters will depend, of course, upon the local requirements for admission.

The present work is a revision and rewriting of a preliminary form which has been in use for three years at the University of Wisconsin. During this time the writer has had frequent and valuable assistance from the instructional force of the department of mathematics in the revision and betterment of the text. Acknowledgments are due especially to Professors Burgess, Dresden, Hart and Wolff and to Instructors Fry, Nyberg and Taylor. Professor Burgess has tested the text in correspondence courses, and has kindly embraced that opportunity to aid very materially in the revision. He has been especially successful in shortening graphical methods and in adapting them to work on squared paper. Professor Wolff has read all of the final manuscript and made many suggestions based upon the use of the text in the class room. Mr. Taylor has read all of the proof and supplied the results to the exercises.

Professor E. V. Huntington of Harvard University has read the galley proof and has contributed many important suggestions.

The writer has avoided the introduction of new technical terms, or terms used in an unusual sense. He has taken the liberty, however of naming the function ax^n, the "Power Function of x," as a short name for this important function seems to be an unfortunate lack—a lack, which is apparently confined solely to the English language.

It is with hesitation that the writer acknowledges his indebtedness to the movement for the improvement of mathematical instruction that has been led by Professor Klein of Göttingen; not that this is not an attempt to produce a text in harmony with that movement, but for fear that the interpretation expressed by the present book is inadequate.

The writer will be glad to receive suggestions from those that make use of the text in the class room.

Charles S. Slichter.

University of Wisconsin
July, 25, 1914

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INTRODUCTION

Any course in mathematics requires the frequent use of geometrical constructions, and the carrying out of analytical and numerical computations. In order that this work may be performed neatly and accurately it is necessary that the student have a few simple instruments, and a supply of proper material for doing the work in a systematic and orderly manner. The indispensible instruments are as follows:

I. Instruments. (1) Two 4*H* hexagonal drawing pencils; one sharpened to a fine point for marking points upon paper or for sketching free hand; the other sharpened to a chisel point for drawing straight lines. Some prefer to use a single pencil sharpened at both ends, one end round pointed, the other end chisel pointed.

(2) A small drawing board[1] of soft wood—10 × 12 inches is large enough.

(3) A small T-square, same length as the drawing board.

(4) A 60° and a 45° transparent triangle. Five-inch triangles are large enough, although a larger 60° triangle will be found to be very convenient.

(5) A protractor for laying off angles.

(6) A triangular boxwood scale, decimally divided.

(7) A pair of 6-inch pencil compasses for drawing circles and arcs of circles, provided with medium hard lead, sharpened to a narrow chisel point.

(8) A 10-inch slide rule is required for Chapter VIII, and may be used earlier at the discretion of the instructor.

[1] Drawing boards of this size with T-square and two wood triangles are marketed by the Milton Bradly Co., Springfield Mass., and by Eugene Dietzgen Co., and Keuffel and Esser of New York and Chicago, and retail for about 40 cents.

II. Materials. All mathematical work should be done on one side of standard size letter paper, $8\frac{1}{2} \times 11$ inches. This is the smallest sheet that permits proper arrangement of mathematical work. There are required:

(1) A notebook cover to hold sheets of the above named size and a supply of manila paper "vertical file folders" for use in submitting work for the examination of the instructor.

(2) A number of different forms of squared paper and computation paper especially prepared for use with this book. These sheets will be described from time to time as needed in the work. Form *M*2 will be found convenient for problem work and for general calculation. *M*2 is a copy of a form used by a number of public utility and industrial corporations. Colleges usually have their own sources of supply of squared paper, satisfactory for use with this book. The forms mentioned in the text, printed on 16 lb., St. Regis Bond, cost about 25 cents per pound in 100 lb. lots (12,000 sheets) from F. C. Blied & Co., Madison, Wis.

(3) Miscellaneous supplies such as thumb tacks, erasers, sandpaper-pencil-sharpeners, etc.

III. General Directions. All drawings should be done in pencil, unless the student has had training in the use of the ruling pen, in which case he may, if he desires, "ink in" the most important drawings.

All mathematical work, such as the solutions of problems and exercises, and work in computation *should be done in ink.* The student should acquire the habit of working problems with pen and ink. He will find that this habit will materially aid him in repressing carelessness and indifference and in acquiring neatness and system.

TO THE INSTRUCTOR

The usual one and one-half year of secondary school Algebra including the solution of quadratic equations and a knowledge of fractional and negative exponents, is required for the work of this course. In the appendix will be found material for a brief review of factoring, quadratics, and exponents, upon which a week or ten days should be spent before beginning the regular work in this text.

The instructor cannot insist too emphatically upon the requirement that all mathematical work done by the student—whether preliminary work, numerical scratch work, or any other kind (except drawings)—shall be carried out with pen and ink upon paper of suitable size. This should, of course, include all work done at home, irrespective of whether it is to be submitted to the instructor or not. The "psychological effect" of this requirement will be found to entrain much more than the acquirement of mere technique. If properly insisted upon, orderly and systematic habits of work will lead to orderly and systematic habits of thought. The final results will be very gratifying to those who sufficiently persist in this requirement.

At institutions whose requirements for admission include more than one and one-half units of preparatory algebra, nearly all of Chapters V, VI, and VII may be omitted from the course.

An asterisk attached to a section number indicates that the section *may be omitted* during the first reading of the book.

GREEK ALPHABET

Capitals	Lower case	Names	Capitals	Lower case	Names
A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ϵ	Epsilon	P	ρ	Rho
Z	ζ	Zeta	Σ	σ	Sigma
H	η	Eta	T	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
I	ι	Iota	Φ	ϕ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

MATHEMATICAL SIGNS AND SYMBOLS

$\dots$	read	*and so on.*
$\equiv$	read	*is identical with.*
$\neq$	read	*is not equal to.*
$\doteq$	read	*approaches.*
$\risingdotseq$	read	*is approximately equal to.*
$>$	read	*is greater than.*
$<$	read	*is less than.*
$\geqq$	read	*is greater than or equal to.*
(a, b)	read	*point whose coördinates are a and b.*
$\lfloor n$	read	*factorial n.*
$n!$	read	*factorial n* or *n admiration.*
$\lim\limits_{x \doteq a} f(x)$	read	*limit of* $f(x)$ as x *approaches* a.
$x = \infty$	read	x *becomes infinite.*
$\|a\|$	read	*absolute value of a.*
$\log_a x$	read	*logarithm of x to the base a.*
$\lg x$	read	*common logarithm of x.*
$\ln x$	read	*natural logarithm of x.*
$\sum\limits_{n=1}^{n=r} u_n$	read	*summation from* $n = 1$ to $n = t$ *of* u_n.

ELEMENTARY MATHEMATICAL ANALYSIS

CHAPTER I

VARIABLES AND FUNCTIONS OF VARIABLES

1. Scales. If a series of points corresponding in order to the numbers of any sequence[1] be selected along any curve, the curve with its points of division is called a **scale.** Thus in Fig. 1 (a) the points along the curve OA have been selected and marked in order with the numbers of the sequence:

$$0,\ 1/4,\ 1/2,\ 1,\ 2\tfrac{1}{2},\ 3,\ 5,\ 7,\ 8.$$

Thus primitive man might have made notches along a twig and then made use of it in making certain measurements of

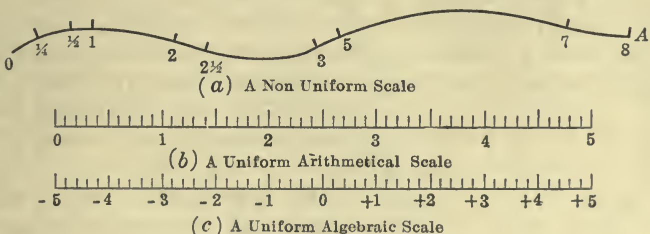


Fig. 1.—Scales of Various Sorts.

interest to him. If such a scale were to become generally used by others, it would be desirable to make many copies of the original scale. It would, therefore, be necessary to use a twig whose shape could be readily duplicated; such, for example, as a straight stick; and it would also be necessary to attach the same symbols invariably to the same divisions.

[1] A sequence of numbers here means a set of numbers arranged in order of magnitude.

Certain advantages are gained (often at the expense of others, however) if the distances between consecutive points of division are kept the same; that is, when the intervals are laid off by repetition of the same selected distance. When this is done, the scale

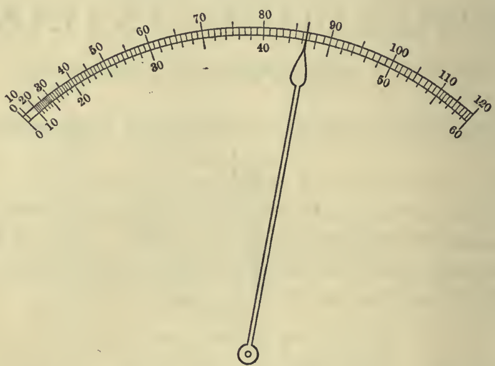


Fig. 2.—An Ammeter Scale.

is called a **uniform scale.** Primitive man might have selected for such uniform distance the length of his foot, or sandal, the breadth of his hand, the distance from elbow to the end of the middle finger (the cubit), the length of a step in pacing (the yard), the amount he can stretch with both arms extended (the fathom), etc., etc.

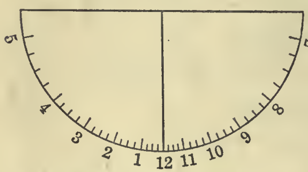


Fig. 3.—Sun-dial Scale

We are familiar with many scales, such as those seen on a yardstick, the dial of a clock, a thermometer, a sun-dial, a steam-gage, an ammeter or voltmeter, the arm of a store-keeper's scales, etc., etc. The scales on a clock, a yardstick, or a steel tape are uniform. Those on a sun-dial, on an ammeter or on a good thermometer, are not uniform.

One of the most important advantages of a uniform scale is the fact that the place of beginning or zero may be taken at any one of the points of division. This is not true of a non-uniform scale. If the needle of an ammeter be bent the instrument cannot be used. It is always necessary in using such an instrument to know that the zero is correct; if a sun-dial is not properly oriented, it is useless. If, however, a yardstick or a steel tape be broken, it may still be used in measuring. The student may think of many other advantages gained in using a uniform scale.

2. Formal Definition of a Scale. If points be selected in order along any curve corresponding, one to one, to the numbers of any sequence, the curve, with its divisions, is called a **scale.**

The notion of **one to one correspondence,** included in this definition, is frequently used in mathematics.

In mathematics we frequently speak of the **arithmetical scale** and of the **algebraic scale.** The arithmetical scale corresponds to the numbers of the sequence:

$$0, 1, 2, 3, 4, 5, \ldots$$

and such intermediate numbers as may be desired. It is usually represented by a uniform scale as in Fig. 1 (b). The algebraic scale corresponds to the numbers of the sequence:

$$\ldots -6, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \ldots$$

and such intermediate numbers as may be desired. It is usually represented by a uniform scale as in Fig. 1 (c). The arithmetical scale begins at 0 and extends indefinitely in one direction. The algebraic scale has no point of beginning; the zero is placed at any desired point and the positive and negative numbers are then attached to the divisions to the right and the left, respectively, of the zero so selected. The scale extends indefinitely in both directions.

Exercises

1. Show that the distance between two points selected anywhern on the algebraic scale is always found by subtraction.

2. If two algebraic scales intersect at right angles, the commoe point being the zero of both scales, explain how to find the distance from any point of one scale to any point of the other scale.

3. What points of the algebraic scale are distant 5 from the point 3 of that scale? What point of the arithmetical scale is distant 5 from the point 3 of that scale?

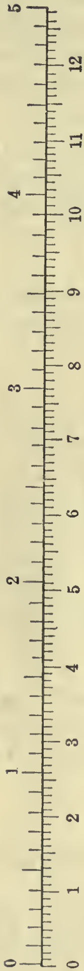


Fig. 4.—Centimeter-Inch Scale

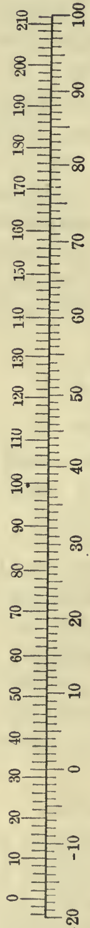


Fig. 5.—Fahrenheit-Centigrade Scale.

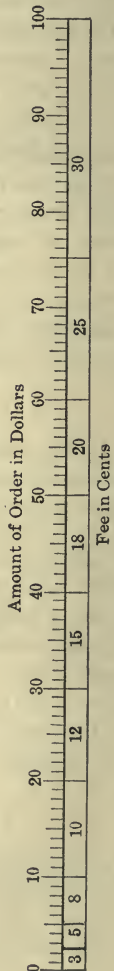


Fig. 6.—Cost of Money Orders.

3. Two Scales in Juxtaposition or Double Scales. The relation between two magnitudes or quantities, or between two numbers, may be shown conveniently by placing two scales side by side. Thus the relation between the number of centimeters and the number of inches in any length may be shown by placing a centimeter scale and a foot-rule side by side with their zeros coinciding as in Fig. 4.

A thermometer is frequently seen bearing both the Fahrenheit and the centigrade scales (see Fig. 5). It is obvious that the double scale of such a thermometer may be used (within the limits of its range) for converting any temperature reading Fahrenheit into the corresponding centigrade equivalent and *vice versa*. The construction of scales of this sort may be made to depend upon the solution of the following problem in elementary geometry: *To divide a given line into a given number of equal parts.*

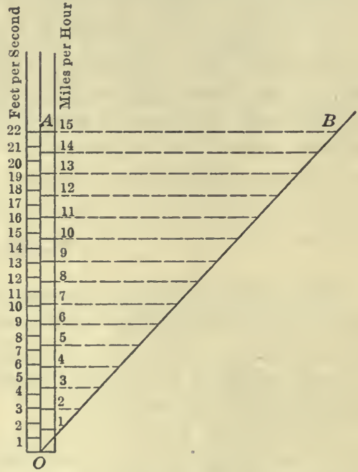


FIG. 7.—Method of Construction of Double Scale showing Relation between "Miles per Hour" and "Feet per Second."

To construct a double scale showing the relation between speed expressed in miles per hour, and speed expressed in feet per second, we may proceed as follows: A mile contains 5280 feet; an hour contains 3600 seconds. Hence, one mile per hour equals 5280/3600 or 22/15 feet per second. On one of two intersecting straight lines, OA (see Fig. 7), lay off 22 convenient equal intervals (say 1/4 inch each). On the second of the intersecting lines, OB, lay off 15 equal intervals (say 1/2 inch each). Join the 15th division of OB with the 22nd division of OA and draw parallels to the line AB through each of the 15 divisions of OB. Then the 22 and the 15 equal subdivisions stand in juxtaposition along OA and constitute the double scale required. Labelling the first scale "feet per second" and the second scale "miles per hour," the double scale may be used for converting speed expressed in either unit into speed expressed in the other.

By annexing the appropriate number of ciphers to the numbers of each scale, the range of the double scale may be considered 220 and 150 or 2200 and 1500, etc., respectively.

The lengths of the various units selected for the diagram are, of course, arbitrary. As, however, the student is expected to prepare the various constructions and diagrams required for the exercises in this book on paper of standard letter size (that is, $8\frac{1}{2}$ by 11 inches), the various units selected should be such as to permit a convenient and practical construction upon sheets of that size.

Exercises

The student is expected to carry out the actual construction of only two of the double or triple scales described in the following exercises.

1. Construct a double scale ten inches long expressing the relation between fractions of an inch expressed in tenths and fractions of an inch expressed in sixteenths.

To draw this double scale it is merely necessary to lay off the intervals directly from suitable foot-rules. On the scale of tenths indicate the inch and half inch intervals by longer division lines than the others. On the scale of sixteenths represent the quarter inch intervals by longer division lines than those of the sixteenths, and represent the half inch and inch intervals by still longer lines, as is usually done on foot rules.

2. Draw a double scale showing pressure expressed as inches of mercury and as feet of water, knowing that the density of mercury is 13.6 times that of water.

These are two of the common ways of expressing pressure. Water pressure at water power plants, and often for city water service, is expressed in terms of head in feet. Barometric pressure, and the vacuum in the suction pipe of a pump and in the exhaust of a condensing steam engine are expressed in inches of mercury. The *approximate relations* between these units, *i.e.*, 1 atmosphere = 30 inches of mercury = 32 feet of water = 15 pounds per square inch, are known to every student of elementary physics. To obtain, in terms of feet of water, the pressure equivalent of 1 foot of mercury, the latter must be multiplied by 13.6, the density of mercury. This result when divided by 12 gives the pressure equivalent of 1 inch of mercury, which is 1.13 feet of water.

If we let the scale of inches of mercury range from 0 to 10, then the scale of feet of water must range from 0 to 11.3. Hence draw a line OA 10 inches long divided into inches and tenths to represent inches of mercury. Draw any line OB through O and lay off 11.3 uniform intervals (inch intervals will be satisfactory) on OB. Connect the end division on OA with the end division on OB by a line AB. Then from 1, 2, 3, . . . inches on OB draw parallels to BA, thus forming adjacent to OA the scale of equivalent feet of water. Each of these intervals can then be subdivided into 10 equal parts corresponding to tenths of feet of water.

3. Draw a triple scale showing pressure expressed as feet of water, as inches of mercury, and as pounds per square inch, knowing that the density of mercury is 13.6 and that one cubic foot of water weighs 62.5 pounds.

To reduce feet of water to pounds per square inch, the weight of one cubic foot of water, 62.5 pounds, must be divided by 144, the number of square inches on one face of a cubic foot. This gives 1 foot of water equivalent to 62.5/144 or 0.434 pounds per square inch. To obtain the pressure given by 1 foot of mercury, the pressure equivalent of 1 foot of water must be multiplied by 13.6, the density of mercury. This result when divided by 12 gives the pressure equivalent of 1 inch of mercury, or 0.492 pounds per square inch.

One pound per square inch is equivalent, therefore, to 1/0.434 or 2.30 feet of water or to 1/0.492 or 2.03 inches of mercury. If we let the scale of pounds range from 0 to 10, we may select 1 inch as the equivalent of 1 pound per square inch, and divide the scale OA into inches and tenths to represent this magnitude. Draw two intersecting lines OB and OC through O, and lay off 23 uniform intervals on OB and lay off 20.3 uniform intervals on OC, 1/2 inch being a convenient length for each of these parts. Connect the end divisions of OB and OC with A and through all points of division of OB draw lines parallel to BA and through all points of division of OC draw lines parallel to CA, and subdivide into halves the intervals of the scales last drawn. The range may be extended to any amount desired by annexing ciphers to the numbers attached to the various scales.

Extending the range by annexing ciphers to the attached numbers is obviously practicable so long as the various intervals or units are decimally subdivided. The method is impracticable for scales that are not decimally subdivided, such as shillings and pence, degrees and minutes, feet and inches, etc.

4. Draw a triple scale showing the relations between the cubic foot, the gallon and the liter, if 1 cubic foot $= 7\frac{1}{2}$ gallons $= 28\frac{1}{3}$ liters. Divide the scale of cubic feet into tenths, the scale of gallons into quarts, and the third scale into liters.

It is obvious that it is always necessary first to select the range of the various scales, but it is quite as well in this case to show the equivalents for 1 cubic foot only, as numbers on the various scales can be multiplied by 10, 100, or 1000, etc., to show the equivalents for larger amounts.

Select 10 inches $=$ 1 cubic foot for the scale (OA) of cubic feet. Draw two intersecting lines OB and OC. On OB lay off $7\frac{1}{2}$ equal parts (say, $7\frac{1}{2}$ inches) and on OC lay off $28\frac{1}{3}$ equal parts (say, $28\frac{1}{3}$ quarter inches). Connect the end divisions with A and draw the parallel lines exactly as with previous examples. The intervals of the scale of gallons can then be subdivided into the four equal parts to show quarts.

5. The velocity in feet per second of a falling body is given by the formula $v = gt$, in which $g = 32.2$ and t is measured in seconds. Draw a double scale showing the velocity at any time.

It is obvious that the reading 32.2 on the v-scale must be placed opposite the mark 1 on the t-scale. First, select the range for the t-scale, say from 1 to 10 seconds. Then a convenient scale for t is 1 inch equals 1 second, which scale can readily be subdivided to show 1/5 or 1/10 seconds. If the general method be followed, it would be necessary to lay off 322 equal parts on a line (OB) intersecting the t-scale (OA). As this is an inconveniently large number, it is better to lay off 3.22 divisions on the construction line OB. Each of these divisions may be 2 inches in length, so that 6.44 inches will represent the terminal or end division on the intersecting line OB. From the 6.44 inch mark on OB draw a line to 10 on the t-scale OA. Then from 2, 4, 6 inches on OB draw parallels to BA, thus locating $v = 100, 200$, and 300. These intervals can then be subdivided into 10 equal parts to show $v = 10, 20, 30, \ldots$ If values of v are wanted for $t > 10$, zeros may be annexed to the numbers attached to both scales.

6. Select sections from any of the double scales described above and discuss the relation of the number of units on one side to the number of units on the other side. Show that the ratio in different sections of the number of units on the two sides of the same double scale is not constant if one scale be a non-uniform scale.

7. If a double scale be drawn on a deformable body, as, for example, on a rubber band, would the double scale still represent true relations when the rubber band is stretched? What if the stretching were not uniform?

4. Functions. The relation between two magnitudes expressed graphically by two scales drawn in juxtaposition, as above, may sometimes be expressed also by means of an equation. Thus, if y is the number of dollars, and x is the number of pounds sterling in any amount, then:

$$y = 4.87\,x \tag{1}$$

also, if F be the reading Fahrenheit, and C the reading centigrade of any temperature, then:

$$F = \tfrac{9}{5}C + 32 \tag{2}$$

also,

$$U = 13.6V/12 = 144W/62.5 \tag{3}$$

where U, V, and W are pressures measured, respectively, in feet of water, inches of mercury, or in pounds per square inch.

NOTE. The letters x, y, F, C, U, V, W in the above equations stand for *numbers;* to make this emphatic we sometimes speak of them as *pure* or *abstract* numbers. These numbers are thought of as arising from the measurement of a magnitude or quantity by the application of a suitable unit of measure. Thus from the magnitude or quantity of water, 12 *gallons*, arises, by use of the unit of measure *the gallon*, the abstract number 12.

Algebraic equations express the relation between numbers, and it should always be understood that the letters used in algebra stand for numbers and not for quantities or magnitudes.

Quantity or **Magnitude** is an answer to the question: "How much?" **Number** is an answer to the question: "How many?"

An interesting relation is given by the scales in Fig. 6. This diagram shows the fee charged for money orders of various amounts; the amount of the order may first be found on the upper scale and then the amount of the fee may be read from the lower scale. The relation here exhibited is quite different from those previously given. For example, note that as the amount of the order changes from \$50.01 to \$60 the fee does not change, but remains fixed at 20 cents. Then as the amount of the order changes from \$60.00 to \$60.01, the fee changes abruptly from 20 cents to 25 cents. For an order of any amount there is a corresponding fee, but for each fee there corresponds not an order of a single value, but orders of a considerable range in value. This is quite different from the cases described in Fig. 5. There for each reading Fahrenheit there corresponds a certain reading centigrade, and *vice versa*, and *for any change, however small*, in one of the temperature readings a change, also small, takes place in the other reading. For this reason the latter quantity is said to be **continuous.**

The relation between the temperature scales has been expressed as an algebraic equation. The relation between the value of a money order and the corresponding fee cannot be expressed by a similar equation. If we had given only a short piece of the centigrade-Fahrenheit double scale, we could, nevertheless, produce it indefinitely in both directions, and hence find the corresponding readings for all desired temperatures. But by knowing the fees for a certain range of money orders one cannot determine the fees for other amounts. In both of these cases, however, we express the fact of dependence of one number upon another number by saying that the *first number is a function of the second number.*

Definition. Any number, u, is said to be a **function** of another number, t, if, when t is given, the value of u is determined. The number t is often called the **argument** of the function u.

Illustrations. The length of a rod is a function of its temperature. The area of a square is a function of the length of a side. The area of a circle is a function of its radius. The square root of a number is a function of the number. The strength of an iron rod is a function of its diameter. The pressure in the ocean is a function of the depth below the surface. The price of a railroad ticket is a function of the distance to be travelled.

It is obvious that any mathematical expression is, by the above definition, a function of the letter or letters that occur in it. Thus, in the equations:

$$u = t^2 + 4t + 1$$

$$u = \frac{t-1}{2t+2}$$

$$u = \sqrt{t+4} + t^2 - \frac{3}{t}$$

u is in each case a function of t.

Temperature Fahrenheit is a function of temperature centigrade. The value of the fee paid for a money order is a function of the amount of the order.

Goods sent by freight are classified into first, second, third, fourth, and fifth classes. The amount of freight on a package is a function of its class. It is also a function of its weight. It is also a function of the distance carried. Only the *second* of these functional relations just named can readily be expressed by an algebraic equation. It is possible, however, to express all three graphically by means of parallel scales. *The definition of the function is given (for any particular railroad) by the complete freight tariff book of the railroad.*

The fee charged for a money order is a function of the amount of the order. The functional relation has been expressed graphically in Fig. 6. Note that for orders of certain amounts, namely, $2½, $5, $10, $20, $30, $40, $50, $60, $75, **the function is not defined.** The graph alone cannot define the function at these values, as one cannot know whether the higher, the lower, or an intermediate fee should be demanded. One can, however, define the function for these values by the supplementary statement (for example): "*For the critical amounts, always charge the higher fee.*" As a matter of fact, however, the lower fee is always charged.

A function having sudden jumps like the one just considered, is said to be **discontinuous.**

Exercises

In the following exercises the function described can be represented by a mathematical expression. The problem is to set up the expression in each case.

1. One side of a rectangle is 10 feet. Express the area A as a function of the other side x.

2. One leg of a right triangle is 15 feet. Express the area A as a function of the other leg x.

3. The base of a triangle is 12 feet. Express the area as a function of the altitude l.

4. Express the circumference of a circle as a function (1) of its radius r; (2) of its diameter d.

5. Express the diagonal d of a square as a function of one side x.

6. One leg of a right triangle is 10. Express the hypotenuse h as a function of the other leg x.

7. A ship B sails on a course AB perpendicular to OA. If $OA = 30$ miles, express the distance of the ship from O as a function of AB.

8. A circle has a radius 10 units. Express the length of a chord as a function of its distance from the center.

9. An isosceles triangle has two sides each equal to 15 cm., and the third side equal to x. Express the area of the triangle as a function of x.

10. A right cone is inscribed in a sphere of radius 12 inches. Express the volume of the cone as a function of its altitude l.

11. A right cone is inscribed in a sphere of radius a. Express the volume of the cone as a function of its altitude l.

12. One dollar is at compound interest for 20 years at r per cent. Express the amount A as a function of r.

Functional Notation. The following notation is used to express that one number is a function of another; thus, if u is a function of t we write:

$$u = f(t)$$

Likewise,

$$y = f(x)$$

means that y is a function of x. Other symbols commonly used to express functions of x are:

$$\phi(x),\ X(x),\ f'(x),\ F(x),\ \text{etc.}$$

These may be read the "ϕ-function of x," the "X-function of x," etc., or more briefly, "the ϕ of x," "the X of x," etc.

Expressing the fact that temperature reading Fahrenheit is a function of temperature reading centigrade, we may write:

$$F = f(C)$$

This is made specific by writing:

$$F = \tfrac{9}{5}C + 32$$

Likewise the fact that the charge for freight is a function of class, weight, and distance, may be written:

$$r = f(c, w, d)$$

To make this functional symbol explicit, might require that we be furnished with the complete schedule as printed in the freight tariff book of the railroad. The dependence of the tariff upon class and weight can usually be readily expressed, but the dependence upon distance often contains arbitrary elements that cause it to vary irregularly, even on different branches of the same railroad. A complete specification of the functional symbol f would be considered given in this case when the tariff book of the railroad was in our hands.

5. Variables and Constants. In elementary algebra, a letter is always used to stand for a number that preserves the same value in the same problem or discussion. Such numbers are called **constants.** In the discussion above we have used letters to stand for numbers that are assumed not to preserve the same value but to change in value; such numbers (and the quantities or magnitudes which they measure) are called **variables.**

If r stands for the distance of the center of mass of the earth from the center of mass of the sun, r is a variable. In the equation $s = \frac{1}{2}gt^2$ (the law of falling bodies), if t be the elapsed time, s the distance traversed from rest by the falling body, and g the acceleration due to gravity, then s and t are variables and g is the constant 32.2 feet per second per second.

The following are constants: Ratio of the diameter to the circumference in any circle; the electrical resistance of pure copper at 60° F.; the combining weight of oxygen; the density of pure iron; the breaking strength of mild steel rods; the velocity of light in empty space.

The following are variables: the pressure of steam in the cylinder of an engine; the price of wheat; the electromotive force in an alternating current; the elevation of groundwater at a given place; the discharge of a river at a given station. When any of these magnitudes are assumed to be measured, the numbers resulting are also *variables.*

The *volume* of the mercury in a common thermometer is a *variable;* the *mass* of mercury in the thermometer is a *constant.*

6. Algebraic Functions. An expression that is built up by operating on x a limited number of times by addition, subtraction, multiplication, division, involution and evolution only, is called an **algebraic function** of x. The following are algebraic functions of x:

(1) x^2. (4) $2x + 5$. (7) $x^3 - 6x^2 + 11x - 6$.

(2) x^n. (5) $1/x$. (8) $\dfrac{x^2 + 3x}{4 + 6x^{1/2}}$

(3) $3\sqrt{x}$. (6) $x^2 - 5$. (9) $(x - a)(x - b)(x - c)$.

The expression x^2 is an algebraic function of x but 2^x is not an algebraic function of x. The fee charged for a money order is not an algebraic function of the amount of the order.

It is convenient to divide algebraic functions into classes. Thus $x^2 + 2$ is said to be **integral**; $(x + 1)/(2 - x^2)$ and $2 + x^{-2}$ are said to be **fractional**; likewise $x^2 + 2$ and $(x + 1)/(2 - x^2)$ are said to be **rational**; $\sqrt{1 - x}$ and $3 - x^{1/2}$ are said to be **irrational.** These terms may be formally defined as follows:

An algebraic function of x is said to be **rational** if in building up the expression, the operation of evolution is not performed upon x, or upon a function of x; otherwise the function is **irrational.**

Thus, expressions (1), (4), (5), (6), (7), (9), above, are rational functions of x. Expressions (3) and (8) are irrational. Expression (2) is rational if n is a whole number; otherwise irrational.

A rational function is said to be **integral** if in building up the function the operation of division by x, or by a function of x, is not performed; otherwise the function is **fractional.**

Thus expressions (1), (4), (6), (7), (9), above, are integral functions of x. Expressions (1), (4), (6), (7), (9) are both rational and integral and may therefore be called **rational integral functions** of x.

Exercises

Classify the following functions of r, t, or x, answering the following questions for each function: (A) is the function algebraic or (B) non-algebraic? If it is algebraic, is it (*a*) rational or (*b*) irrational; if it is rational, is it (1) integral or (2) fractional? The scheme of classification is as follows:

A. Algebraic.

$$(a)\ \text{rational} \begin{cases} (1)\ \text{integral} \\ (2)\ \text{fractional} \end{cases}$$

(*b*) irrational

B. Non-algebraic.

1. $16.1t^2$; $\sqrt{a^2 - x^2}$; $\sqrt{ax^4}$; $\sqrt{a/x}$.

2. $ax^3 + bx^2 + cx + d.$

3. $x^{\frac{2}{3}} - x^{\frac{1}{2}}.$

4. $x^3 + \frac{1}{x} + x^{\frac{1}{2}}.$

5. $2x + x^2 + 2^x + \frac{2}{x}.$

6. $\frac{1+t}{1-t}$; $(1 + t)(1 - t)$; $(1 + \sqrt{t})(1 - \sqrt{t})$.

7. $mx + \sqrt{a^2 - x^2}$; $3.37x^{1.86}\,t^{1.25}$.

8. $\frac{a^2 - x^2}{a + x}$; $\frac{a^3 - x^3}{a - x}$; $\frac{a^3 + x^3}{a + x}$.

9. $(a - x)(a^2 + ax + x^2)$; $(a^{\frac{1}{3}} - x^{\frac{1}{3}})(a^{\frac{2}{3}} + a^{\frac{2}{3}}x^{\frac{1}{3}} + x^{\frac{1}{3}})$.

10. $\frac{a - ar^n}{1 - r}$. Write an equal integral expression.

7. Graphical Computation. The ordinary operations of arithmetic, such as multiplication, division, involution and evolution,

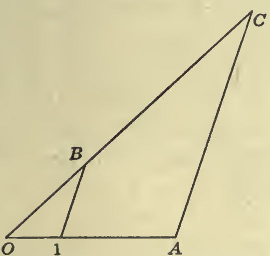


FIG. 8.—Graphical Multiplication by Properties of Similar Triangles.

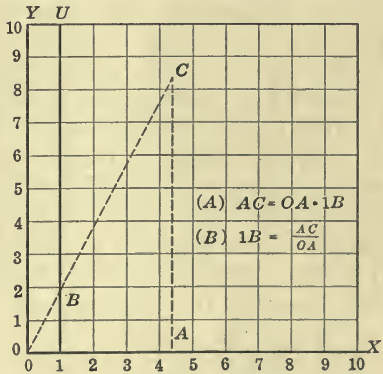


FIG. 9.—Method of Graphical Multiplication and Division carried out on Squared Paper. The figure shows 1.9 × 4.4 = 8.4.

can be performed graphically as explained below. The graphical construction of products and quotients is useful in many problems of science. The law of proportional sides of similar triangles is the fundamental theorem in all graphical computation. Its application is very simple, as will appear from the following work.

PROBLEM 1: *To compute graphically the product of two numbers.* Let the two numbers whose product is required be a and b. On any line lay off the unit of measurement, $O1$, Fig. 8. On the same line, and, of course, to the same scale, lay off OA equal to one of the factors a. On any other line passing through 1 lay off a line $1B$ equal to the other factor b. Join OB and produce it to meet AC drawn parallel to $1B$. Then AC is the required product. For, from similar triangles:

$$AC : 1B = OA : O1 \tag{1}$$

or,

$$\mathbf{AC = OA \times 1B} \tag{A}$$

It is obvious that the angle OAC may be of any magnitude. Hence it may conveniently be taken a right angle, in which case the work may readily be carried out on ordinary squared paper. Many prefer, however, to do the work on plain paper, laying off the required distances by means of a boxwood triangular scale. The squared paper, form $M1$, prepared for use with this book is suitable for this purpose. On a sheet of this paper, draw the two lines OX and OY at right angles and the unit line $1U$, as shown in Fig. 9. Then from the similar triangles $O1B$ and OAC the proportion (1) and the formula (A) above are true. Hence to compute graphically the product of two numbers a and b count off (Fig. 9) $OA = a$ to the OX-scale and $1B = b$ to the OY-scale. Lay a straight edge or edge of a transparent triangle down to draw OC. It is not necessary to draw OC, but merely to locate the point C. Then count off AC to the OY-scale. Then $AC = a \times b$ by (A). The figure as drawn shows the product $4.4 \times 1.9 = 8.4$.

All numbers can be multiplied graphically on a section of squared paper 10 units in each dimension by properly reading the OX and OY scales. Any product ab can be written $a_1b_1 \times 10^n = c_1 \times 10^n$, where a_1 and b_1 each have one digit before the decimal point, and $c_1 \leq 100$.

Thus:

$$440 \times 19 = 4.40 \times 1.9 \times 10^3 = 8.40 \times 10^3$$

also

$$37 \times 73 = 3.7 \times 7.3 \times 10^2 = 27 \times 10^2$$

To proceed with the product of $a_1 \times b_1$, we first determine by inspection whether $c_1 >$ or < 10. If $c_1 < 10$, we read the scales as they are in Fig. 9 when counting off a_1, b_1 and c_1. If $c_1 > 10$, we read the OX scale as it stands when counting off a_1 but read the OY scale 0, 10, 20, 30, etc., in counting off the numbers b_1 and c_1.

Exercises

In using form $M1$ for the following exercises take the scale OY at the left marginal line of the sheet and use 2 cm. as the unit of measure. Compute graphically the following products: Check results:

1. 2.5×4.8.	**4.** 78.5×16.5.
2. 4.15×6.25.	**5.** 2.14×0.0467.
3. 3.14×7.22.	**6.** 2140×0.0467.

PROBLEM 2: *To compute graphically the quotient of two numbers a and b.* Formula (A) above can be written:

$$1B = \frac{AC}{OA} \tag{B}$$

From this it is seen that the quotient of two numbers a and b can readily be computed graphically by use of Figs. 8 or 9. In Fig. 9 count off $OA = b$, the divisor, to the OX scale, and $AC = a$, the dividend, to the OY scale. Lay the triangle down to draw OC. Do not draw OC, but mark the point B and count off $1B$ to the OY scale. Then $1B = a/b$ by (B). Fig. 9 shows the quotient $8.4 \div 4.4 = 1.9$. Any quotient a/b may be written

$$\frac{N}{D} \times 10^n = Q \times 10^n$$

where N, D, Q are each $\leqq 10$ but > 1. Hence, the OX and OY scales may always be read as they stand in Fig. 9.

Exercises

Compute graphically the following quotients: Check results:

1. $6.2/2.5$.	**4.** $7.32/1.25$.
2. $1.33/6.45$.	**5.** $872/321$.
3. $234/0.52$.	**6.** $128/937$.

PROBLEM 3: *To compute graphically the square root of any number* N. In Fig. 10 count off $1A = N$ to the OX scale, and draw a semicircle on OA as a diameter. Then $1C = \sqrt{N}$ to the OY scale. Another construction is to place the triangle in the position shown in Fig. 10, so that the two edges pass through O and A and the vertex of the right angle lies on the line $1U$. Fig. 10 shows the construction for $\sqrt{7}$. The readings on the OX

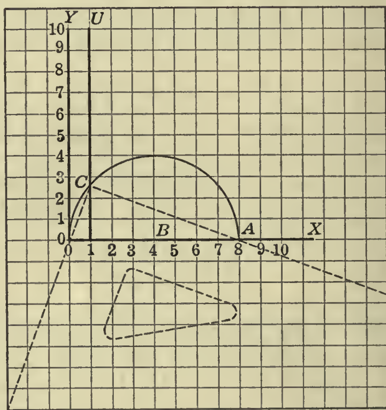


FIG. 10.—Graphical Method of the Extraction of Square Roots. The figure shows $\sqrt{7} = 2.65$.

scale may be multiplied by 10^{2n} and those on the OY scale by 10^n where n is any integer positive or negative.

State the two theorems in plane geometry on which the proof of these two constructions depends.

PROBLEM 4: *To compute graphically the square of any number* N. This is a special case of Problem 1, when $a = b = N$.

Exercises

1. Compute the square roots of 2, 3, 5, and 7.
2. Compute the square roots of 3.75, 37.5, 0.375.
3. Compute the squares of 1.23 and 3.45.
4. Compute the squares of 7.75 and 0.895.
5. Show that π^2 is nearly 10.

PROBLEM 5: *To compute graphically the reciprocal of any number N.* This is a special case of Problem 2, when $a = 1$ and $b = N$.

PROBLEM 6: *To compute graphically the integral powers of any number N.* This problem is solved by the successive application of Problem 1 to construct N^2, N^3, N^4, etc., and of Problem 2

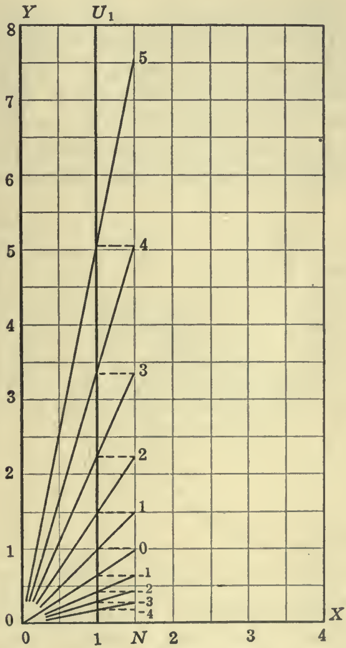


FIG. 11.—Graphical Computation of $(1.5)^n$ for $n = -4, -3, -2, -1$ $0, 1, 2, 3, 4, 5$.

to construct N^{-1}, N^{-2}, N^{-3}, etc. This construction is shown for the powers of 1.5 in Fig. 11.

Exercises

1. Compute the reciprocal of 2.5; of 3.33; of 0.75; of 7.5.

2. Compute $(1.2)^3$, $(0.85)^3$, $(1.15)^4$.

3. Show that $(1.05)^{15} = 2.08$, so that money at 5 percent compound interest more than doubles itself in fifteen years.

NOTE: The work is less if $(1.05)^5$ is first found and then this result cubed.

4. From the following outline the student is to produce a complete method, including proof, of constructing successive powers of any number.

Let OA (Fig. 12) be a radius of a circle whose center is O. Let OB be any other radius making an acute angle with OA. From B drop a perpendicular upon OA, meeting the latter at A_1. From A_1 drop a perpendicular upon OB meeting OB at A_2. From A_2 drop a perpendicular upon OA meeting OA at A_3, and so on indefinitely. Then, if OA be unity, OA_1 is less than unity, and OA_2, OA_3, OA_4 . . . are, respectively, the square, cube, fourth power, etc., of OA_1.

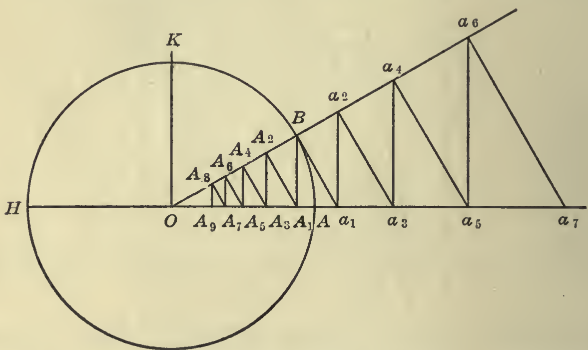


FIG. 12.—Graphical Computation of Powers of a Number.

Instead of the above construction, erect a perpendicular to OB meeting OA produced at a_1. At a_1 erect a perpendicular meeting OB produced at a_2, and so on indefinitely. Then if OA be unity, a_1 is greater than unity and $a_2, a_3, a_4,$ . . . are, respectively, the square, cube, etc., of a_1. As an exercise, construct powers of 4/5 and of 2.5.

5. Show that the successive "treads and risers" of the steps of the "stairways" of Figs. 13 and 14 are proportional to the powers of r. The figures are from Milaukovitch, *Zeitschrift für Math. und Nat. Unterricht*, Vol. 40, p. 329.

8. Double Scales for Several Simple Algebraic Functions. We may make use of the graphical method of computation explained above to construct graphically double scales representing simple algebraic relations. For example, we may construct a double scale for determining the square of any desired number.

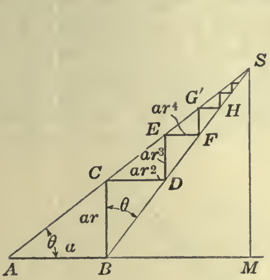


FIG. 13.

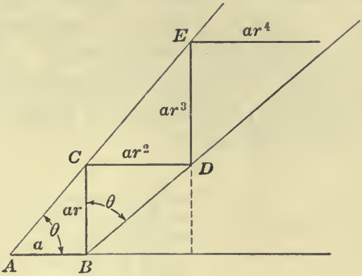


FIG. 14.

Computation of ar, ar^2, ar^3, . . . for $r < 1$ and for $r > 1$.

Call OA (see Fig. 15) the scale on which we desire to read the number; call OB the scale on which we read the square. Let us agree to lay off OA as a uniform scale, using $O1$ as the unit of measure. Since we desire to read opposite 0, 1, 2, 3, of the

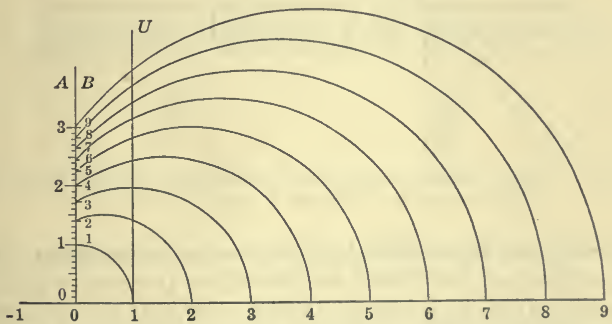


FIG. 15.—Method of Constructing a Double Scale of Squares or of Square Roots.

uniform scale, the squares of these numbers, the lengths along the scale OB must be laid off proportional to the *square roots* of the numbers 0, 1, 2, 3, . . . , that is, the square root of any length, when laid off on OB, *and marked with the symbol of the original length, will be opposite the square root of that number on OA.*

No difficulty need be experienced in carrying out the actual construction of double scales representing algebraic relations, either by use of a table of numerical values of the function or by means of graphical construction. As a less laborious method of graphically expressing functional relations will be explained in the next chapter, the matter of double scales will not be discussed further at this place.

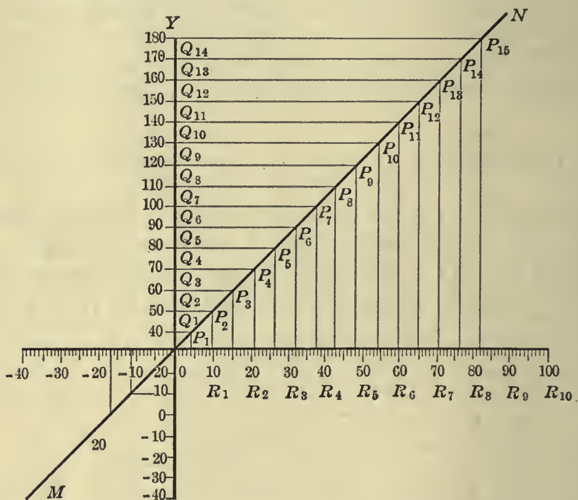


FIG. 16.—The Fahrenheit-centigrade Double Scale Opened about the 32° Mark of the Fahrenheit Scale as Pivot.

9. Functions Represented by Scales not in Juxtaposition. It is obvious that any double scale used to express the relation between a function of a variable and the variable itself, may be separated, if desired, into two distinct scales, provided means be adopted for connecting corresponding points on the two scales. For example, one of the two scales may be rotated about any one of its points, as scissors about their pivot, thereby forming two intersecting straight lines. Corresponding points may then be connected by erecting perpendiculars to each scale and joining those that proceed from corresponding points, or by any other practical means. In Fig. 16 the centigrade and Fahrenheit scales are shown opened about the 32° division of the Fahrenheit scale as pivot. Perpendiculars erected at corresponding points of the two scales meet at the points P_1, P_2, P_3, . . .

The line NOM on which these points lie is straight. Why? The student will write out a proof, making use of any three points as P_1, P_2, P_3, and a property of similar triangles. Of course the angle between OC and OF need not be taken as a right angle.

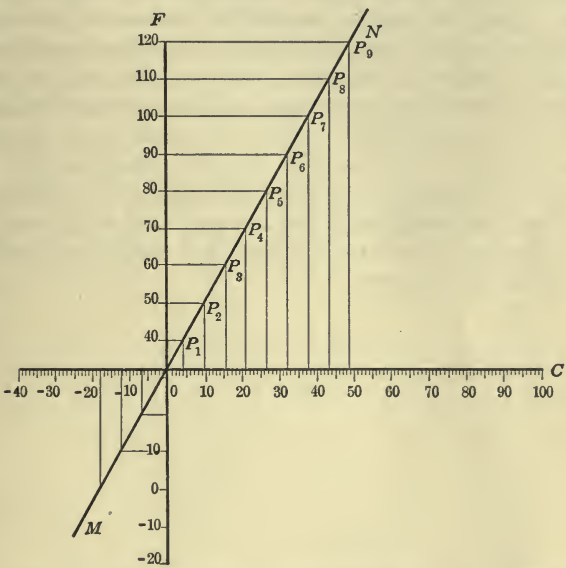


FIG. 17.—Same as Fig. 16 with the Lengths of the Units on the OC and OF Scales Made the Same.

It is also obvious that the divisions on both scales may now be made the same length; that is, OQ_1, OQ_2, OQ_3, . . . may be made the same length as OR_1, OR_2, OR_3, ... This is at once accomplished if the lines OQ_1, OQ_2, OQ_3, . . . , be each elongated in the ratio of OR_1/OQ_1. The functional relation may be expressed equally well by marking as before the intersection of the perpendiculars erected at corresponding values. The result is shown in Fig. 17.

In the same manner any of the double scales may be opened about any point as pivot. If the angle between the scales is made 90°, the relation between the function and its argument is shown by points on a straight line making an angle of 45° with each scale. If one of the scales be non-uniform, it may, after it is turned about the selected pivot, be made a uniform scale, in which case the straight line just mentioned becomes, in general, a curved line. We see, therefore, that instead of showing the relation between a function and its variable by means of two scales in juxtaposition, we may use two uniform scales intersecting at an angle, and connect corresponding values of the variable and its function by perpendiculars erected at these corresponding points. The pairs of perpendiculars intersect at points which, in general, lie upon a curve. This curve is obviously characteristic of the particular functional relation under discussion. The respresentation of functional relations in this manner leads to the consideration of so-called coördinate systems, the discussion of which is begun in the next chapter.

CHAPTER II

RECTANGULAR COÖRDINATES AND THE POWER FUNCTION

10. Rectangular Coördinates. Two intersecting algebraic scales, with their zero points in common, may be used as a system of latitude and longitude to locate any point in their plane. The student should be familiar with the rudiments of this method from the graphical work of elementary algebra. The scheme is illus-

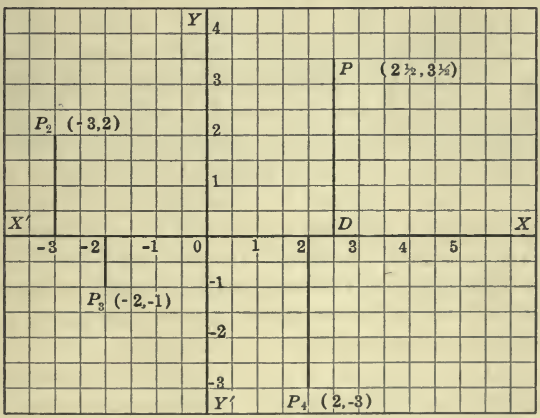


FIG. 18.—Rectangular Coördinates.

trated in its simplest form in Fig. 18, where one of the horizontal lines of a sheet of squared paper has been selected as one of the algebraic scales and one of the vertical lines of the squared paper has been selected for the second algebraic scale. To locate a given point in the plane it is merely necessary to give, in a suitable unit of measure (as centimeter, inch, etc.), the distance of the point to the right or left of the vertical scale and its distance above or below the horizontal scale. Thus the point P, in Fig. 18, is $2\frac{1}{2}$ units to the right and $3\frac{1}{2}$ units above the standard scales. P_2 is 3 units to the left and 2 units above the standard scales, etc. Of course these directions are to be given in mathematics by the use of the signs "$+$" and "$-$" of the algebraic scales, and not by the use of the words "right" or "left," "up" or "down." The above scheme corresponds to the location of a place on the earth's surface by giving its angular distance in degrees of longitude east or west of the standard meridian, and also by giving its angular distance in degrees of latitude north or south of the equator.

The sort of latitude and longitude that is set up in the manner described above is known in mathematics as **a system of rectangular coördinates.** It has become customary to letter one of the scales XX', called the **X-axis,** and to letter the other YY', called the **Y-axis.** In the standard case these are drawn to the right and left, and up and down, respectively, as shown in Fig. 18. The distance of any point from the Y-axis, measured parallel to the X-axis, is called the **abscissa** of the point. The distance of any point from the X-axis, measured parallel to the Y-axis, is called the **ordinate** of the point. Collectively, the abscissa and ordinate are spoken of as the **coördinates** of the point. **Abscissa** corresponds to the **longitude** and **ordinate** corresponds to the **latitude** of the point, referred to the X-axis as equator, and to the Y-axis as standard meridian. In the standard case, abscissas measured to the right of YY' are reckoned positive, those to the left, negative. Ordinates measured up are reckoned positive, those measured down, negative.

Rectangular coördinates are frequently called **Cartesian coördinates,** because they were first introduced into mathematics by René Descartes (1596–1650).

The point of intersection of the axes is lettered O and is called the **origin.** The four quadrants, XOY, YOX', $X'OY'$, $Y'OX$, are called the *first, second, third,* and *fourth* **quadrants,** respectively.

A point is designated by writing its abscissa and ordinate in a parenthesis and in this order: Thus, (3, 4) means the point whose abscissa is 3 and whose ordinate is 4. Likewise $(-3, 4)$ means the point whose abscissa is (-3) and whose ordinate is $(+4)$.

Unless the contrary is explicitly stated, the scales of the coördinate axes are assumed to be *straight* and *uniform* and to intersect at right angles. Exceptions to this are not uncommon, however, of which examples are given in Figs. 19 and 22.

The use of two intersecting algebraic scales to locate individual points in the plane, as explained above, is capable of immediate enlargement. It will be explained below that a suitable array, or set, or locus of such points may be used to exhibit the relation between two variables laid off on the two scales, or between a variable laid off on one of the scales and a function of the variable laid off on the other scale. This fact has already been explained from another point of view at the close of the preceding chapter.

11. Statistical Graphs. From work in elementary algebra the student is supposed to be familiar with the construction of statis-

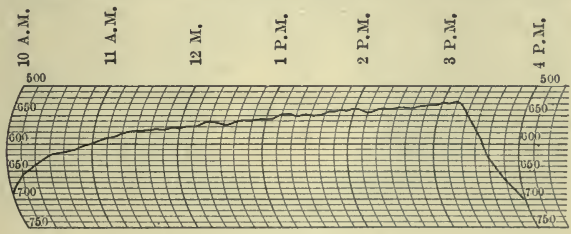


Fig. 19.—Barograph Taken During a Balloon Journey. The vertical scale is atmospheric pressure in millimeters of mercury.

tical graphs similar to those presented in Figs. 19 to 32. The student will study each of these graphs and the following brief descriptions before making any of the drawings required in the exercises that follow.

Fig. 19 is a barograph, or autographic record of the atmospheric pressure recorded November 24, 1907, during a balloon journey from Frankfort to Marienburg in West Prussia. One set of scales consists of equal circles, the other of parallel straight lines. The zero of the scale of pressure does not appear in the diagram. Note also that the scale of pressure is an *inverted* scale, increasing downward. The scale of time is an algebraic scale, the zero of which may be arbitrarily selected at any convenient point. The scale of pressure is an arithmetical scale. The zero of the barometric scale corresponds to a perfect vacuum—no less pressure exists.

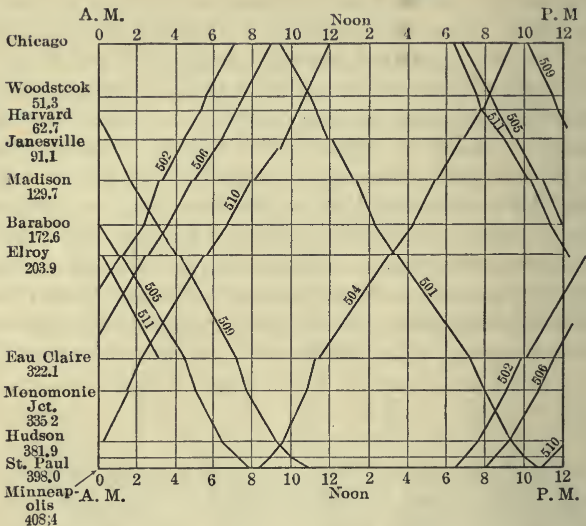


FIG. 20.—Graphical Time-table of Certain Railway Trains between Chicago and Minneapolis.

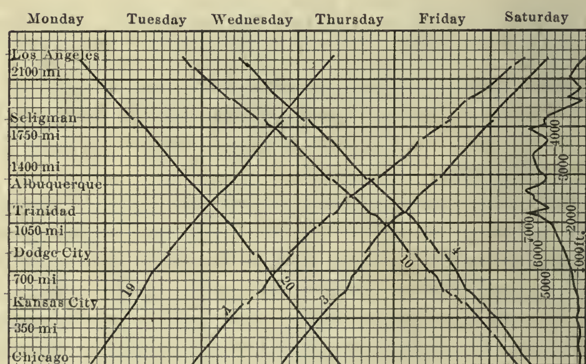


FIG. 21.—Graphical Time-table of Passenger Trains between Chicago and Los Angeles.

Fig. 20 is a graphical time-table of certain passenger trains between Chicago and Minneapolis. The curves are not continuous, as in the case of the barograph, but contain certain sudden jumps. What is the meaning of these? What indicates the speed of the

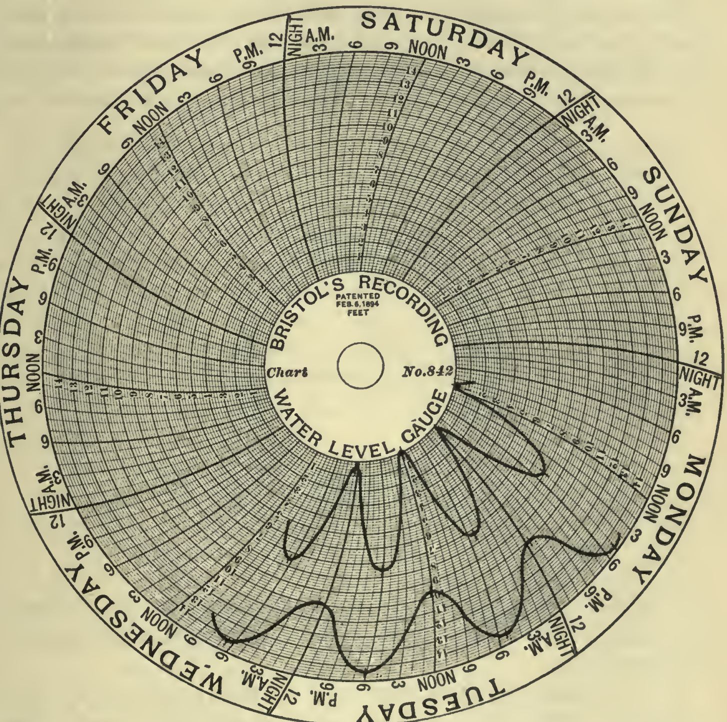


Fig. 22.—Upper Curve, Elevation of Water in a Well on Long Island Lower curve, elevation of water in the nearby ocean.

trains? Where is the fastest track on this railroad? What shows the *meeting point* of trains?

If the diagram, Fig. 20, be wrapped around a vertical cylinder of such size that the two midnight lines just coincide, then each train line may be traced through continuously from terminus to terminus.

Functions having this remarkable property are said to be **periodic.** In the present case the trains run at the same time every day, that is, periodically. In mathematical language, the position of the trains is said to be a **periodic function of the time.**

Fig. 21 is the graphical time-table of "limited" trains between Chicago and Los Angeles. The schedule of train No. 1, a very heavy passenger train, is placed upon the chart for comparison. The periodic character of this function is brought out very clearly by using time as the abscissa. The student should discuss the

FIG. 23.—The Graph of a Discontinuous Function.

discontinuities and the various speeds as shown from the diagram. The track profile is given at the right of the diagram for purposes of comparison.

Fig. 22 represents the fluctuation of the elevation of the ground-water at a certain point near the sea-coast on Long Island. The fluctuations are primarily due to the tidal wave in the near-by ocean. Here the scale of one of the coördinates (elevation) is laid off on a series of equal circumferences similar to those of Fig. 19. The scale of the other coordinate (time) is laid off on the margin of the outer or bounding circle. The curve is continuous. Is the curve periodic? What indicates the *rate of change* in the elevation of the ground-water? When is the elevation changing most rapidly? When is it changing most slowly?

Fig. 23 represents the functional relation between the amount of a domestic money order and the fee. Two arithmetical scales were used in making the diagram, as in ordinary rectangular coördinates, except that the vertical scale is ten-fold the horizontal scale; that is, lengths that represent dollars on the one scale represent cents on the other. This is an excellent illustration of a *discontinuous* function. On account of the sudden jumps in the values of the fee, the fee, as explained in the preceding chapter, is said to be a *discontinuous function* of the amount of the order.

12. Suggestions on the Construction of Graphs. Two kinds of rectangular coördinate paper have been prepared for use with this book. Form $M1$ is ruled in centimeters and fifths, and permits two scales of twenty and twenty-five major units respectively to be laid off horizontally and vertically on a standard sheet of letter paper $8\frac{1}{2} \times 11$ inches. Form $M2$ is ruled without major divisions in uniform 1/5-inch intervals. This form of ruling is desirable for general computation and for graphing functions for which non-decimal fractional intervals are used, such as eighths, twelfths, or sixteenths, which often occur in the measurement of mass or time.

It is a mistake to assume that more accurate work can be done on finely ruled than on more coarsely ruled squared paper. Quite the contrary is the case. Paper ruled to 1/20-inch intervals does not permit interpolation within the small intervals while paper ruled to 1/10 or 1/5-inch intervals permits accurate interpolation to one-tenth of the smallest interval. Form $M1$ is ruled to 2-mm. intervals, and is fine enough for any work. The centimeter unit has the very considerable advantage of permitting *twenty* of the units within the width of an ordinary sheet of letter paper ($8\frac{1}{2} \times 11$ inches) while *seven* is the largest number of inch units available on such paper.

In order to secure satisfactory results, the student must recognize that there are several varieties of statistical graphs, and that each sort requires appropriate treatment.

1. It is possible to make a useful graph when only *one* variable is given. Thus the following table gives the ultimate tensile strength of various materials:

ULTIMATE TENSILE STRENGTH OF VARIOUS MATERIALS

Material	Tensile strength, tons per square inch
Hard steel..................	50.0
Structural steel..............	30.0
Wrought iron................	25.0
Drawn brass................	21.5
Drawn copper...............	16.0
Cast brass..................	12.0
Cast copper.................	11.0
Cast iron...................	10.0
Timber, with grain..........	5.0

A graph showing these results is given in Fig. 24. There are two practical ways of showing the numerical values pertaining to each material, both of which are indicated in the diagram; either rectangles of appropriate height may be erected opposite the name of each material, or points marked by circles, dots or crosses may be located at the appropriate height. It is obvious in this case that a smooth curve should *not* be drawn through these points —such a curve would be quite meaningless. In this case there are not two scales, but merely the single vertical scale. The horizontal axis bears merely the *names* of the different materials and has no *numerical* or *quantitative* significance. The result is obviously not the graph of a function, for there are not two variables, but only one. The graph is merely a convenient expression for certain discrete and independent results arranged in order of descending magnitude.

2. It is possible to have a graph involving two variables in which it is either impossible or undesirable to represent the graph by a continuous curve or line. For example, Fig. 25 is a graph representing the maximum temperature on each day of a certain month. Because there is only one maximum temperature on each day, the value corresponding to this should be shown by an appropriate rectangle, or by marking a point by a circle, or by a dot or cross, as in the preceding case, since a continuous curve through these points has no meaning. The horizontal scale may

be marked by the *names* of the days of the week or by *numbers*, but in either case the horizontal line is a true scale, as it corresponds to the lapse of the variable time. Sometimes, as in Fig. 25, graphs of this kind are represented by marking the appropriate points by dots or circles and then connecting the successive points by straight lines. These lines have no special meaning in such a case, but they aid the eye in following the succession of separate points.

If a graph be made of the noonday temperatures of each day of the same month referred to in Fig. 25, one of the same methods indicated above would be used to represent the results; that is, either rectangles, marked points, or marked points joined by lines. Although a smooth curve drawn through the known points would have a meaning (if correct), it is obvious that the noonday temperatures alone are not sufficient for determining its form. In all such cases a smooth curve should not be drawn.

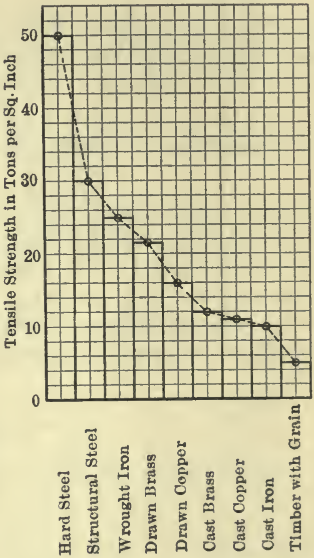


FIG. 24.—Graph Showing Tensile Strength of Certain Structural Materials.

Fig. 26 shows the monthly output and gross earnings of a power company during its first months of operation; the fixed charges are also shown upon the same diagram. (See also Figs. 24 and 84.)

3. If the data are reasonably sufficient, a smooth curve may, and often should, be drawn through the known points. Thus if the temperature be observed every hour of the day and the results be plotted, a smooth curve drawn carefully through the known points will probably very accurately represent the unknown temperatures at intermediate times. The same may safely be done in exercises (1) and (2) below. In scientific work it is desir-

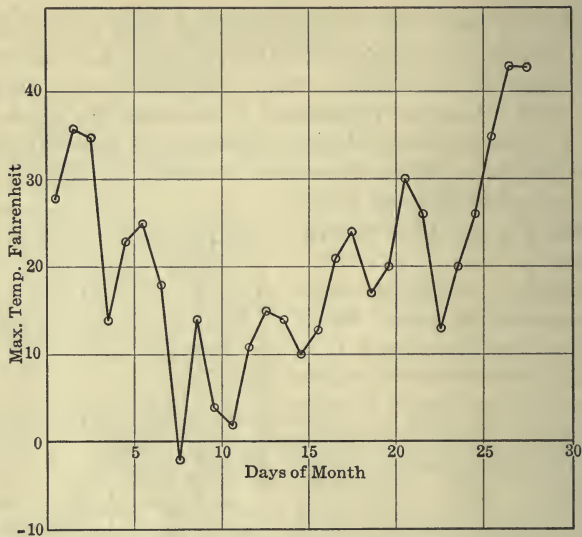


Fig. 25.—Maximum Daily Temperatures, Madison, Wis., February, 1914

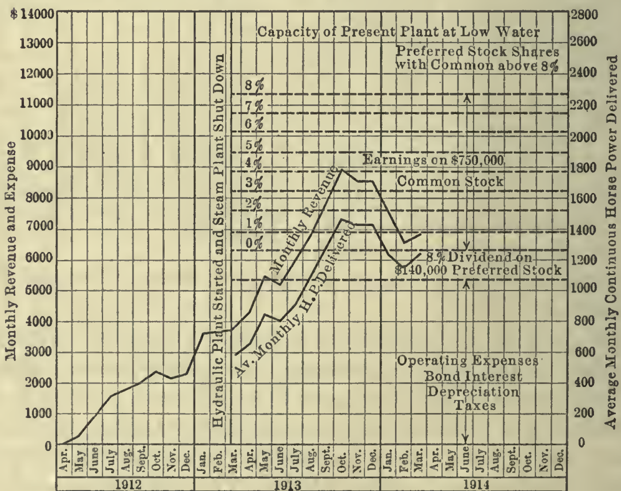


Fig. 26.—Graph of Monthly Gross Earnings and Output of a Power Plant During Initial Stages of Operation.

able to mark by circles or dots the values that are actually given to distinguish them from the intermediate values "guessed" and represented by the smooth curve.

In addition to the above suggestions, the student should adhere to the following instructions:

4. Every graph should be marked with suitable numerals along both numerical scales.

5. Each scale of a statistical graph should bear in words a description of the magnitude represented and the name of the unit of measure used. These words should be printed in drafting letters and not written in script.

6. Each graph should bear a suitable title telling exactly what is represented by the graph.

7. The selection of the units for the scale of abscissas and ordinates is an important practical matter in which common sense must control. It is obvious that in the first exercise given below 1 cm. = 1 foot draft for the horizontal scale, and 1 cm. = 100 tons for the vertical scale will be units suitable for use on form $M1$.

Further instruction in practical graphing is given in §33.

Exercises

1. At the following drafts a ship has the displacements stated:

Draft in feet, h	15	12	9	6.3
Displacement in tons, T.....	2096	1512	1018	586

Plot on squared paper. What are the displacements when the drafts are 11 and 13 feet, respectively?

2. The following tests were made upon a steam turbine generator:

Output in kilowatts, K........	1,190	995	745	498	247
Weight, pounds of steam consumed per hour, W.	23,120	20,040	16,630	12,560	8,320

Plot on squared paper. What are the probable values of K when W is 22,000 and also when W is 11,000?

3. Make a graphical chart of the zone rates of the Parcel Post Service for the first three zones, using weight of package as abscissa and cost of postage as ordinate.

4. The average temperature at Madison from records taken at 7 a. m. daily for 30 years is as follows:

Jan. 1, 14.0. F.	July 1, 67.5. F.
Feb. 1, 15.1.	Aug. 1, 64.0.
Mar. 1, 35.2.	Sept. 1, 55.4.
Apr. 1, 40.0.	Oct. 1, 44.1.
May 1, 53.9.	Nov. 1, 30.0.
June 1, 63.2.	Dec. 1, 18.3.

Make a suitable graph of these results on squared paper.

13. Mathematical, or Non-statistical Graphs. Instead of the expressions "*abscissa of a point,*" or "*ordinate of a point,*" it has become usual to speak merely of the **"x of a point,"** or of the **"y of a point,"** since these distances are conventionally represented by the letters x and y, respectively. If we impose certain conditions upon x and y, then it will be found that we have, by that very fact, restricted the possible points of the plane located by them to a certain array, or set, or locus of points, and that all other points of the plane fail to satisfy the conditions or restrictions imposed.

It is obvious that the command, "Find the place whose latitude equals its longitude," does not restrict or confine a person to a particular place or point. The places satisfying this condition are unlimited in number. We indicate all such points by drawing a line bisecting the angles of the first and third quadrants; at all points on this line latitude equals longitude. We speak of this line as the **locus** of all points satisfying the conditions. We might describe the same locus by saying "the y of each point of the locus equals the x," or, with the maximum brevity, simply write the equation "$y = x$." This is said to be the **equation of the locus,** and the line is called the **locus of the equation.**

It is of the utmost importance to be able readily to interpret any condition imposed upon, or, what is the same thing, any relation between variables, when these are given in words. It will greatly aid the beginner in mastering the concept of what is meant by the term *function* if he will try to think of the *meaning in words* of the relations commonly given by equations, and *vice versa.* The very elegance and brevity of the mathematical expression of relations by means of equations, tends to make work with them formal and mechanical unless care is taken by the beginner to express in words the ideas and relations so briefly expressed by the equations. Unless expressed in words, the ideas are liable not to be expressed at all.

The **equation of a curve** is an equation satisfied by the coördinates of every point of the curve and by the coördinates of no other point.

The **graph of an equation** is the locus of a point whose coördinates satisfy the equation.

Exercises

1. Draw and discuss the following loci:

The ordinate of any point of a certain locus is twice its abscissa; the x of every point of a certain locus is half its y; the y of a point is 1/3 of its x; a point moves in such a way that its latitude is always treble its longitude; the sum of the latitude and longitude of a point is zero; a point moves so that the difference in its latitude and longitude is always zero.

2. Draw this locus: Beginning at the point (1, 2), a point moves so that its gain in latitude is always twice as great as its gain in longitude.

3. A point moves so that its latitude is always greater by 2 units than three times its longitude. Write the equation of the locus and construct.

4. A head of 100 feet of water causes a pressure at the bottom of 43.43 pounds per square inch. Draw a locus showing the relation between head and pressure, for all heads of water from 0 to 200 feet.

Suggestion: There are several ways of proceeding. Let *pounds per square inch* be represented by abscissas or x, and *feet of water* be represented by ordinates or y. Then we take the point $x = 43.43$, $y = 100$ and other points, as $x = 86.86$, $y = 200$, etc., and draw the line. Otherwise produce the equation first from the proportion $x:y::43.43:100$, or, $43.43y = 100x$ or $y = \dfrac{100}{43.43}x$ and then draw the graph from the fact that the latitude is always $\dfrac{100}{43.43}$ of the longitude.

Be sure that the scales are numbered and labeled in accordance with suggestions (4), (5) and (6) of §**12.**

5. A pressure of 1 pound per square inch is equivalent to a column of 2.042 inches of mercury, or to one of 2.309 feet of water. Draw a locus showing the relation between pressure expressed in feet of water and pressure expressed in inches of mercury.

Suggestion: Let x = inches of mercury and y = feet of water. First properly number and label the X-axis to express inches of mercury and number and label the Y-axis to express feet of water. Since negative numbers are not involved in this exercise, the origin may be taken at the lower left-hand corner of the squared paper. First locate the point $x = 2.042$, $y = 2.309$ (which are the corresponding values given by the problem) and draw a line through it and the origin. This is the required locus since at all points we must have the proportion $x : y :: 2.042 : 2.309$, which says that the ordinate of every point of the locus is 2309/2042 times the abscissa of that point.

6. A certain mixture of concrete (in fact, the mixture 1 : 2 : 5) contains 1.4 barrels of cement in a cubic yard of concrete. Draw a locus showing the cost of cement per cubic yard of concrete for a range of prices of cement from \$0.80 to \$2.00 per barrel.

Suggestion: Let x be the price per barrel of cement and y be the cost of the cement in 1 cubic yard of concrete. Number and label the two scales beginning at the lower left-hand corner as origin. Since prices between \$0.80 and \$2.00 only need be considered, the first division on the X-axis may be marked \$0.80 instead of 0. Each centimeter may represent \$0.10 on each scale. The cost of cement per cubic yard of concrete must, by the condition of the problem, be 1.4 times the price per barrel of cement. Hence the first point located on the vertical scale must correspond to $1.4 \times \$0.80$, or to \$1.12 cost per cubic yard. As this is the lowest cost to be entered, it is desirable not to start the vertical scale at \$0.00, but at \$1.00. Thus the lower left-hand corner of the coördinate paper may be taken as the point (0.80, 1.00) in a system in which the unit of measure is 1 cm. = 10 cents.

7. Draw a locus showing the cost per cubic yard of concrete for various prices of cement, provided \$2.10 per yard must be added to the results of example 6 to cover cost of sand and crushed stone.

8. Cast iron pipe, class A (for heads under 100 feet), weighs, per foot of length: 4-inch, 20.0 pounds; 6-inch, 30.8 pounds; 8-inch, 42.9 pounds. For each size of pipe construct upon a single sheet of squared paper a locus showing the cost per foot for all variations in market price between $20.00 and $40.00 per ton.

SUGGESTION: If the horizontal scale be selected to represent *price per ton*, the scale may begin at 20 and end at 40, as this covers the range required by the problem. Therefore let 1 cm. represent $1.00. Since the range of prices is from 1 cent to 2 cents per pound, the cost per foot will range from 20 cents to 40 cents for 4-inch pipe and from 42.9 cents to 85.8 cents for 8-inch pipe. Hence for the vertical scale 10 cents may be represented by 2 cm. In this case the vertical scale may quite as well begin at 0 cents instead of at 20 cents, as there is plenty of room on the paper.

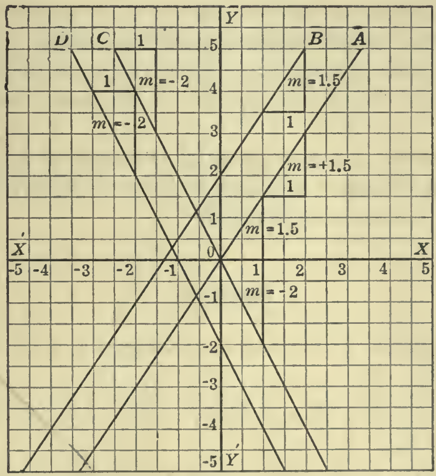


FIG. 27.—Lines of Slope (1.5) and of Slope (−2).

14. Slope. The **slope** of a straight line is defined to be the **change** in y for an **increase** in x equal to 1. It will be represented in this book by the letter m. Thus in Fig. 27 the line A has the slope $m = 1.5$, for it is seen that at any point of the line the ordinate y gains 1.5 units for an increase of 1 in x. The line B, parallel to the line A, is also seen to have the slope equal to 1.5. The equation of the line A is obviously $y = 1.5x$. In the same figure the slope of the line C is -2, for at any point of this line the ordinate y *loses* 2 units for an increase in x equal to 1. The equation of the line C is obviously $y = -2x$. Line D, parallel to line C, also has slope (-2)

If h be the change in y for an increase of x equal to k, then the slope m is the ratio h/k.

The technical word *slope* differs from the word *slope* or *slant* in common language only in the fact that slope, in its technical use, is always expressed as a ratio. In common language we speak of a "slope of 1 in 10," or a "grade of 50 feet per mile," etc. In mathematics the equivalents are "slope = 1/10," "slope = 50/5280," etc.

As already indicated, the definition of slope requires us to speak in mathematics of **positive slope** and **negative slope.** A line of positive slope extends upward with respect to the standard direction OX and a line of negative slope extends downward with reference to OX.

In a similar way we may speak of the **slope of any curve** at a given point on the curve, meaning thereby the slope of the tangent line drawn to the curve at that point.

Exercises

1. Give the slopes of the lines in exercises 1 to 8 of the preceding set of exercises.

2. Draw $y = x$; $y = 2x$; $y = 3x$; $y = \frac{2x}{3}$; $y = \frac{x}{2}$; $y = \frac{x}{4}$; $y = -2x$; $y = -3x$; $y = 0x$.

3. Prove that $y = mx$ always represents a straight line, no matter what value m may have.

15. Equation of Any Line. Intercepts.—In Fig. 28, the line MN expresses that the ordinate y is, for all points on the line, always 3 times the abscissa x, or it says that $y = 3x$. The line HK states that "y is 2 more than $3x$." Thus the line HK has the equation $y = 3x + 2$.

In general, since $y = mx$ is always a straight line,[1] then $y = mx + b$ is a straight line, for the y of this locus is merely, in each case, the y of the former increased by the constant amount b (which

[1] See exercise 3, §14, above.

may, of course, be positive or negative). Therefore, $y = mx + b$ is a line parallel to $y = mx$. The distance OB (Fig. 28) is equal to b. The distance is called the **y-intercept** of the locus. The distance OA is equal to $-b/m$, for it is the value of x when y is zero. It is called the **x-intercept** of the locus.

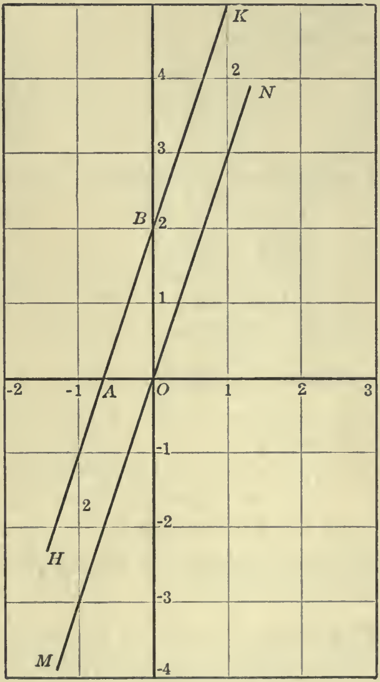


FIG. 28.—Intercepts.

Exercises

1. Sketch, from inspection of the equations, the lines given by:

(a) $y = x$.	(d) $y = x + 3$.
(b) $y = x + 1$.	(e) $y = x - 1$.
(c) $y = x + 2$.	(f) $y = x - 2$.

2. Sketch, from inspection of the equations, the lines given by:

(*a*) $y = \frac{1}{2}x$.
(*b*) $y = \frac{3}{4}x$.
(*c*) $y = x$.
(*d*) $y = 2x$.
(*e*) $y = 3x$.
(*f*) $y = -\frac{1}{2}x$.
(*g*) $y = -x$.
(*h*) $y = -2x$.
(*i*) $y = -3x$.
(*j*) $y = \sqrt{2}x$.

3. Sketch the lines given by:

(*a*) $x = 3$.
(*b*) $x = 5$.
(*c*) $x = -2$.
(*d*) $y = 1$.
(*e*) $y = 5$.
(*f*) $y = -3$.
(*g*) $y = 0$.
(*h*) $x = 0$.
(*i*) $x^2 = 4$.

4. Sketch from inspection of the equations, the following:

(*a*) $y = x + 1$.
(*b*) $y = \frac{1}{2}x + 1$.
(*c*) $y = -2x + 4$.
(*d*) $y = 5x + 3$.
(*e*) $y = -5x - 2$.

5. Sketch, from inspection of the equations:

(*a*) $y = x + 4$.
(*b*) $y - 2x - 3 = 0$.
(*c*) $y + \frac{2}{3}x + 1/3 = 0$.
(*d*) $ax + by = c$.
(*e*) $x/a + y/b = 1$.

6. The shortest distance between $y = mx$ and $y = mx + b$ is not b. Show that it equals $b/\sqrt{1 + m^2}$.

16. Additive Properties. Sometimes a useful result is obtained by adding (or subtracting) the corresponding ordinates of two graphs. Thus in Fig. 26, operating expenses of a power plant may be added to ordinates representing various rates of dividends, and compared (by subtraction) with monthly revenue. Sometimes, however, it becomes necessary to determine a result by adding two functions corresponding to *different* values of the variable or argument. Fig. 29 is an excellent illustration of this. This diagram enables one to find the cost of a cubic yard of "1:2:4" concrete (except cost of mixing) by knowing the prices of the constituent materials. The information necessary to construct the loci is given in the first line of Table I, p. 44. The amount of cement in 1 cubic yard of 1:2:4 concrete is seen to be 1.58 barrels. The price per barrel of cement may be considered a variable changing with the condition of the market and with the locality where sold. Calling x_1 the price of cement, the cost y_1 of the cement in 1 cubic yard of 1:2:4 concrete is then, for all market prices of cement, expressed by the equation:

$$y_1 = 1.58x_1$$

This is graphically represented in Fig. 29 by the line of slope 1.58. Note in this case that the slope of the line has a "physical" meaning, namely it is the *cost* of the cement in 1 cubic yard when the *price* is $1.00 a barrel. In the same way the cost of the sand and of the crushed stone in 1 cubic yard of concrete for various market prices of these commodities is expressed by the lines of Fig. 29 of slopes 0.44 and 0.88 respectively.

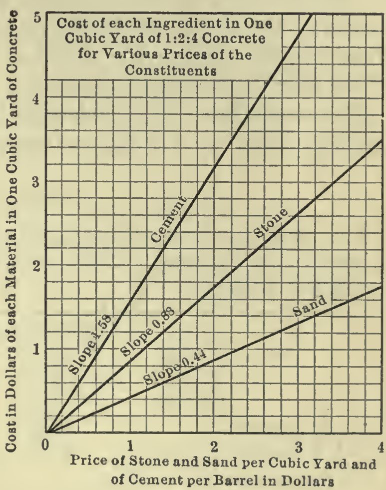


Fig. 29.

Example: Let the price x_1 of cement be $1.20 per barrel; let the price x_2 of stone be $1.75 per cubic yard, and the price x_3 of sand be $1.10 per cubic yard. Find the cost of the materials necessary to make 1 cubic yard of 1:2:4 concrete. Then, from Fig. 29:

$$\begin{aligned} x_1 &= \$1.20 \text{ then } & y_1 &= \$1.90 \\ x_2 &= \ \ 1.75 & y_2 &= \ \ 1.54 \\ x_3 &= \ \ 1.10 & y_3 &= \ \ \underline{0.48} \end{aligned}$$

Total, or cost of material for 1 cubic yard of concrete $= \$3.92$

The cost of concrete, y, is a function of three variables, x_1, x_2,

x_3, all of which, for convenience' sake, have been measured on the same scale or axis OX. The representation of several variables on the same scale need not cause any confusion.

Since in this case the prices of the constituents of the concrete are not the same, the total cost of 1 cubic yard of concrete cannot be found by adding the ordinates at the same abscissa of the three graphs, because the abscissas or the various market prices of the ingredients are not the same.

The second line of the table may be used by the student as the basis of construction of another diagram similar to that of Fig. 29.

TABLE I

The quantities of material required to make 1 cubic yard of concrete (based on $33\frac{1}{3}$ percent voids in the sand and 45 percent voids in the broken stone).

Mixture	Quantities of materials in 1 cubic yard. of concrete		
	Cement, barrels*	Sand, cubic yards	Stone, cubic yards
1:2:4 concrete.........	1.58	0.44	0.88
1:2½:5 concrete........	1.33	0.46	0.92
1:3:6 concrete.........	....	....	....

* A barrel (4 bags) of cement weighs 380 pounds and contains $3\frac{4}{5}$ cubic feet of cement.

NOTE: The student may be interested to know how the figures in the first line of the table are obtained. The explanation will best be understood if the figures as given are first verified. First the 1.58 barrels of cement should be reduced to cubic yards. It gives 0.22 cubic yard. A part of this must be used to fill the $33\frac{1}{3}$ percent of voids in the 0.44 cubic yard of sand. The cement required for this is 0.146 cubic yard. Thus the sand and cement combine to make $0.44 + 0.22 - 0.146$ or 0.514 of mixed material. A part of this mixture is used to fill the 45 percent voids in the 0.88 cubic yard of stone, which equals 0.396 cubic yard. Hence the total volume of stone, sand and cement is $0.88 + 0.514 - 0.396$, which equals 0.998, or the cubic yard required.

To find the numbers in the table, the above process needs to be reversed and stated algebraically. Thus, to make a cubic yard of 1:2:4 concrete let

$$x = \text{cubic yards cement required.}$$
$$y = \text{cubic yards sand required.}$$
$$z = \text{cubic yards stone required.}$$

Then, from the given porosities, or percent of voids,

$$x - \tfrac{1}{3}y = \text{surplus of cement after filling voids in sand.}$$
$$(x - \tfrac{1}{3}y) + y = \text{volume of mixed sand and cement.}$$
$$[(x - \tfrac{1}{3}y) + y] - 0.45z = \text{surplus of mixed sand and cement after filling voids in stone.}$$
$$z + [(x - \tfrac{1}{3}y) + y] - 0.45z = 1, \text{ the total volume,}$$

or,

$$0.55z + \tfrac{2}{3}y + x = 1$$

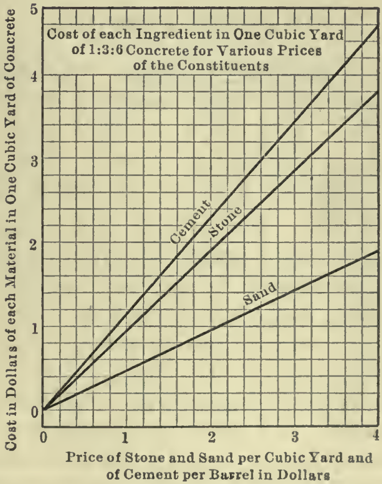


Fig. 30.

Also, because the mixture is 1:2:4:

$$2x = y$$
$$4x = z$$

These give:

$$4.53x = 1$$
$$x = 0.22 \text{ cubic yard}$$

Shorter reasoning is as follows: As the voids in the crushed stone are to be completely filled in the finished concrete, the z cubic yards of stone counts as only $0.55 \cdot z$ cubic yards in the final product. As the voids in the sand are to be completely filled in the final mixture, the y cubic yards of sand counts as only $\frac{2}{3} \cdot y$ cubic yards in the final product. As there are no voids to be filled in the cement, it counts as x cubic yards in the final result. Hence the equation

$$x + \tfrac{2}{3}y + 0.55z = 1, \text{ etc.}$$

Exercise

From the diagram, Fig. 30, determine and insert in a table like Table I, the quantity of each sort of material in 1 cubic yard of 1 : 3 : 6 concrete.

THE POWER FUNCTION

17. Definition of the Power Function. The algebraic function consisting of a single power of the variable, such for example as the functions x^2, x^3, $1/x$, $1/x^2$, $x^{2/3}$, etc., stand next to the linear function of a single variable, $mx + b$, in fundamental importance. The function x^n is known as the **power function of x.**

18. The Graph of x². The variable part of many functions of practical importance is the **square** of a given variable. Thus the area of a circle depends upon the *square* of the radius; the distance traversed by a falling body depends upon the square of the elapsed time; the pressure upon a flat surface exposed directly to the wind depends upon the square of the velocity of the wind; the heat generated in an electric current in a given time depends upon the square of the number of amperes of current, etc., etc. Each of these relations is expressed by an equation of the form $y = ax^2$, in which x stands for the number of units in one of the variable quantities (radius of the circle, time of fall, velocity of the wind, amperes of current, respectively, in the above named cases) and in which y stands for the other variable dependent upon these. The number a is a constant which has a value suitable to each particular problem, but in general is not the same constant in different problems. Thus, if y be taken as the area of a circle, $y = \pi x^2$, in which x is the radius measured in feet or inches, etc., and y is measured in square feet or square inches, etc.; or if s is the distance in feet traversed by a falling body, then $s = 16.1t^2$, where t stands for the elapsed time in seconds. In one case the value of the constant a is 3.1416 and in the other its value is 16.1.

Let us first graph the abstract law or equation $y = x^2$, in which a concrete meaning is not assumed for the variables x and y but in which both are thought of as abstract variables. First form a suitable table of values for x and x^2 as follows:

x	-3	-2	-1	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2	3
x^2 or y	9	4	1	0	0.04	0.16	0.36	0.64	1.0	1.44	1.96	2.56	3.24	4	9

Here we have a series of pairs of values of x and y which are associated by the relation $y = x^2$. Using the x of each pair of values as abscissa with its corresponding y there can be located as many points as there are pairs of values in the table, and the array of points thus marked may be connected by a freely drawn curve. To draw the curve upon coördinate paper, form $M1$, the origin may be taken at the mid-point of the sheet, and 2 cm. used as the unit of measure for x and y. If the points given by the pairs of values are not located fairly close together, it is obvious that a smooth curve cannot be satisfactorily sketched between the points until intermediate points are located by using intermediate values of x in forming the table of values. The student should think of the curve as extending indefinitely beyond the limits of the sheet of paper used; the entire locus consists of the part actually drawn and of the endless portions that must be followed in imagination beyond the range of the paper. If the graph of $y = x^2$ be folded about the Y-axis, OY, it will be noted at once that the left and right portions of the curve will exactly coincide. The student will explain the reason for this fact.

19. Parabolic Curves. The equations $y = x$, $y = x^2$, $y = x^{\frac{3}{2}}$, $y = x^3$ should be graphed by the student on a single sheet of coördinate paper, using 2 cm. as the unit of measure in each case. Table II may be used to save numerical computation in the construction of the graphs of these power functions. As in the case of $y = x^2$, a smooth curve should be sketched free-hand through the points located by means of the table of values, and intermediate values of x and y should be computed when doubt exists in the mind of the student concerning the course of the curve between any two points.

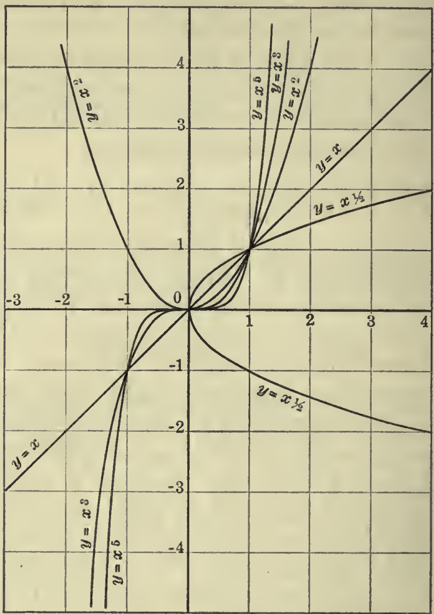


FIG. 31.—Parabolic Curves.

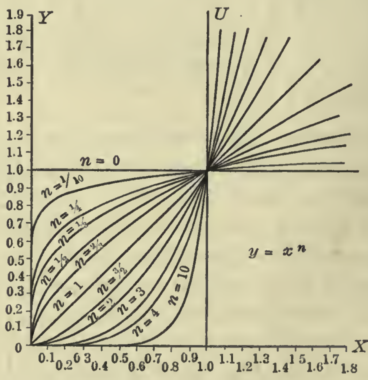


FIG. 32.—Graph of the Power Function for $n > 0$ (Parabolic Curves) in the First Quadrant.

The graphs of the above power functions are observed to be *continuous lines*, without breaks or sudden jumps. A formal proof

TABLE II

x	x^2	x^3	$\sqrt{x}$	$\sqrt[3]{x}$	$x^{3/2}$	$1/x$	$1/x^2$
0.2	0.04	0.008	0.447	0.585	0.089	5.000	25.000
0.4	0.16	0.064	0.632	0.737	0.252	2.500	6.250
0.6	0.36	0.216	0.775	0.843	0.465	1.667	2.778
0.8	0.64	0.512	0.894	0.928	0.715	1.250	1.563
1.0	1.00	1.000	1.000	1.000	1.000	1.000	1.000
1.2	1.44	1.728	1.095	1.063	1.312	0.8333	0.6944
1.4	1.96	2.744	1.183	1.119	1.657	0.7143	0.5102
1.6	2.56	4.096	1.265	1.170	2.034	0.6250	0.3906
1.8	3.24	5.832	1.342	1.216	2.415	0.5556	0.3086
2.0	4.00	8.000	1.414	1.260	2.828	0.5000	0.2500
2.2	4.84	10.65	1.483	1.301	3.263	0.4545	0.2066
2.4	5.76	13.82	1.549	1.339	3.717	0.4167	0.1736
2.6	6.76	17.58	1.612	1.375	4.193	0.3846	0.1479
2.8	7.84	21.95	1.673	1.409	4.685	0.3571	0.1276
3.0	9.00	27.00	1.732	1.442	5.196	0.3333	0.1111
3.2	10.24	32.77	1.789	1.474	5.724	0.3125	0.0977
3.4	11.56	39.30	1.844	1.504	6.269	0.2941	0.0865
3.6	12.96	46.66	1.897	1.533	6.831	0.2778	0.0772
3.8	14.44	54.87	1.949	1.560	7.407	0.2632	0.0693
4.0	16.00	64.00	2.000	1.587	8.000	0.2500	0.0625
4.2	17.64	74.09	2.049	1.613	8.608	0.2381	0.0567
4.4	19.36	85.18	2.098	1.639	9.229	0.2273	0.0517
4.6	21.16	97.34	2.145	1.663	9.866	0.2174	0.0473
4.8	23.04	110.6	2.191	1.687	10.42	0.2083	0.0434
5.0	25.00	125.0	2.236	1.710	11.18	0.2000	0.0400
5.2	27.04	140.6	2.280	1.732	11.85	0.1923	0.0370
5.4	29.16	157.5	2.324	1.754	12.66	0.1852	0.0343
5.6	31.36	175.6	2.366	1.776	13.25	0.1786	0.0319
5.8	33.64	195.1	2.408	1.797	13.97	0.1724	0.0297

TABLE II.—(*Continued*)

x	x^2	x^3	$\sqrt{x}$	$\sqrt[3]{x}$	$x^{3/2}$	$1/x$	$1/x^2$
6.0	36.00	216.0	2.449	1.817	14.70	0.1667	0.0278
6.2	38.44	238.3	2.490	1.837	15.44	0.1613	0.0260
6.4	40.96	262.1	2.530	1.857	16.19	0.1563	0.0244
6.6	43.56	287.5	2.569	1.876	16.96	0.1515	0.0230
6.8	46.24	314.4	2.608	1.895	17.33	0.1471	0.0216
7.0	49.00	343.0	2.646	1.913	18.52	0.1429	0.0204
7.2	51.84	373.2	2.683	1.931	19.32	0.1389	0.0193
7.4	54.76	405.2	2.720	1.949	20.13	0.1351	0.0183
7.6	57.76	439.0	2.757	1.966	20.95	0.1316	0.0173
7.8	60.84	474.6	2.793	1.983	21.79	0.1282	0.0164
8.0	64.00	512.0	2.828	2.000	22.63	0.1250	0.0156
8.2	67.24	551.4	2.864	2.017	23.48	0.1220	0.0149
8.4	70.56	592.7	2.898	2.033	24.35	0.1190	0.0142
8.6	73.96	636.1	2.933	2.049	25.22	0.1163	0.0135
8.8	77.44	681.5	2.966	2.065	26.11	0.1136	0.0129
9.0	81.00	729.0	3.000	2.080	27.00	0.1111	0.0123
9.2	84.64	778.7	3.033	2.095	27.91	0.1087	0.0118
9.4	88.36	830.6	3.066	2.110	28.82	0.1064	0.0113
9.6	92.16	884.7	3.098	2.125	29.74	0.1042	0.0109
9.8	96.04	941.2	3.130	2.140	30.68	0.1020	0.0104
10.0	100.00	1000.0	3.162	2.154	31.62	0.1000	0.0100

that x^n is a continuous function for any positive, rational value of n will be given later.

All of the graphs here considered have one important property in common, namely, they all pass through the points (0, 0) and (1, 1). It is obvious that this property may be affirmed of any curve of the class $y = x^n$, *if n is a positive number*. These curves are known collectively as *curves of the parabolic family*, or simply **parabolic curves.** The curve $y = x^2$ is called **the parabola.** $y = x^3$ is called the **cubical parabola.** $y = x^{3/2}$ is called the **semi-cubical parabola,** etc. Curves for negative values of n do not pass through the point (0, 0) and are otherwise quite distinct. They are known as curves of the **hyperbolic type,** and will be discussed later.

The student should cut patterns of the parabola, the cubical parabola and the semi-cubical parabola out of heavy paper for use in drawing these curves when required. Each pattern should have drawn upon it either the x- or y-axis and one of the unit lines to assist in properly adjusting the pattern upon squared paper.

20. Symmetry. In geometry a distinction is made between two kinds of symmetry of plane figures—symmetry with respect to a *line* and symmetry with respect to a *point.* A plane figure is symmetrical with respect to a given line if the two parts of the figure exactly coincide when folded about that line. Thus the letters **M** and **W** are each symmetrical with respect to a vertical line drawn through the vertex of the middle angles. We have already noted that $y = x^2$ is symmetrical with respect to OY.

A plane figure is symmetrical with respect to a given point when the figure remains unchanged if rotated 180° in its own plane about an axis perpendicular to the plane at the given point. Thus the letters **N** and **Z** are each symmetrical with respect to the mid-point of their central line. The letters **H** and **O** are symmetrical both with respect to lines and with respect to a point. Which sort of symmetry is possessed by the curve $y = x^3$? Why?

Another definition of symmetry with respect to a point is perhaps clearer than the one given in above statement: A curve is said to be symmetrical with respect to a given point O when all lines drawn through the given point and terminated by the curve are bisected at the point O.

What kind of symmetry with respect to one of the coördinate axes or to the origin (as the case may be) does the point $(2, 3)$ bear to the point $(-2, 3)$? To the point $(-2, -3)$? To the point $(2, -3)$?

Note that symmetry of the first kind means that a plane figure is unchanged when turned 180° about *a certain line in its plane,* and that symmetry of the second kind means that a figure is unchanged when turned 180° about a certain line *perpendicular to its plane.*

21. The curves in the diagram, Fig. 31, are sketched from a limited number of points only, but any number of additional values of x and y may be tabulated and the accuracy, as well as the extent, of the graph be made as great as desired. A number of graphs of power functions are shown as they appear in the *first quadrant* in Figs. 32 and 35. The student should explain how to draw the portions of the curves lying in the other quadrants from the part appearing in the first quadrant.

In the exercises in this book to "draw a curve" means to construct the curve as accurately as possible from numerical or other data. To "sketch a curve" means to produce an approximate or less accurate representation of the curve, including therein its characteristic properties, but without the use of extended numerical data.

Exercises

1. On cöordinate paper draw the curves $y = x^2$, $y = x^3$, $y = x^{3/2}$, $y = x^5$, using 4 cm. as the unit of measure. On the same sheet draw the lines $x = \pm 1$, $y = \pm 1$, $y = \pm x$.

2. On coördinate paper sketch the curves $x = y^2$, $x = y^3$, $x = y^{3/2}$, $x = y^5$. Compare with the curves of exercise 1.

3. Sketch and discuss the curves $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = \sqrt[4]{x}$. Can any of these curves be drawn from patterns made from the curves of exercise 1? Why? Explain the graphs of the first and last if the *double sign* "$\pm$" be understood before the radicals, and compare with the graphs when the positive sign only is to be understood before the radicals.

4. Draw the curve $y^2 = x^4$. Compare with the curve $y = x^2$.

5. Name in each case the quadrants of the curves of exercises 1–4, and state the reasons why each curve exists in certain quadrants and why not in the other quadrants.

22. Discussion of the Parabolic Curves. Draw the straight lines $x = 1, x = -1, y = 1, y = -1$ upon the same sheet upon which a number of parabolic curves have been drawn. These lines together with the coördinate axes divide the plane into a number of rectangular spaces. In Fig. 33 these spaces are shown divided into two sets, those represented by the cross-hatching, and those shown plain. *The cross-hatched rectangular spaces contain the lines $y = x$ and $y = -x$ and also all curves of the parabolic type. No parabolic curve ever enters the rectangular strips shown plain in* Fig. 33.

The line $y = x$ divides the spaces occupied by the parabolic curves into equal portions. Why does the curve $y = x^2$ (in the first quadrant) lie below this line in the interval $x = 0$ to $x = 1$, but above it in the interval to the right of $x = 1$? On the other hand, why does the curve $y = \sqrt{x}$, or $y^2 = x$ (in the first quadrant), lie above the line $y = x$ in the interval $x = 0$ to $x = 1$ and below $y = x$ in the interval to the right of $x = 1$?

One part of the parabolic curve $y = x^n$ always lies in the first quadrant. If n be an even number, another part of the curve lies

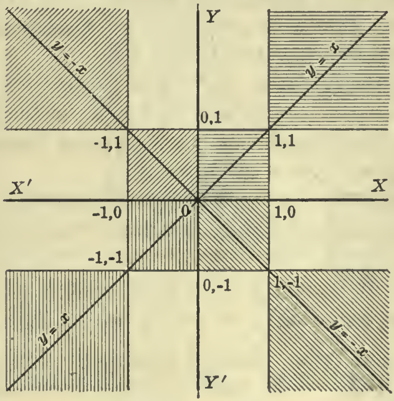


FIG. 33.—The Regions of the Parabolic and the Hyperbolic Curves. All parabolic curves lie within the cross-hatched region. All hyperbolic curves lie within the region shown plain.

in which quadrant? If n be an odd number, the curve lies in which quadrants?

If the exponent n of any power function be a positive fraction, may m/r, the equation of the curve may be written $y^r = x^m$. If in this case both m and r be odd, the curve lies in which quadrants? If m be even and r be odd, the curve lies in which quadrants? If m be odd and r be even, the curve lies in which quadrants? If both m and r be even the curve lies in which quadrants?

A curve which is *symmetrical to another curve with respect to a line* may be spoken of figuratively as the **reflection** or **image** of the second curve in a mirror represented by the given line.

Exercises

Exercises 1–5 refer to curves in the first quadrant only.

1. The expressions x^2, $x^{3/2}$, x^3, x^5 are numerically less than x for values of x between 0 and 1. How is this fact shown in the diagram, Fig. 31?

2. The expressions x^2, $x^{3/2}$, x^3, x^5 are numerically greater than x for all values of x numerically greater than unity. How is this fact pictured in the diagram, Fig. 31?

3. For values of x between 0 and 1, $x^5 < x^3 < x^2 < x^{3/2} < x$. For values $x > 1$, $x^5 > x^3 > x^2 > x^{3/2} > x$. Explain how each of these facts is expressed by the curves of Fig. 32.

4. Show that the graphs $y = x^{2/3}$, $y = x^{1/2}$, $y = x^{1/3}$, $y = x^{1/5}$ are the reflections of $y = x^{3/2}$, $y = x^2$, $y = x^3$, $y = x^5$, in the mirror $y = x$.

5. Sketch without tabulating the numerical values, the following loci: $y = x^{10}$, $y = x^{0.1}$, $y = x^{100}$, $y = x^{0.01}$.

The following are to be discussed for all quadrants.

6. Sketch, without tabulating numerical values, the following loci $y^2 = x^4$, $y^4 = x^6$, $y^4 = x^2$, $y^3 = x^5$, $y^5 = x^3$.

7. Sketch the following: $y^{99} = x^{101}$, $y^{101} = x^{99}$, $y^{1000} = x^{1001}$.

8. Sketch the following: $y = -x^2$, $y = -x^3$, $y^2 = -x^3$.

23. Hyperbolic Type. Loci of equations of the form $yx^n = 1$, or $y = 1/x^n$, where n is positive, have been called **hyperbolic curves.** The fundamental curve $xy = 1$, or $y = 1/x$ is called the **rectangular hyperbola.** Its graph is given in Figs. 34 and 35, but the curve should be drawn independently by the student, using 2 cm. as the unit of measure. Its relation to the x- and y-axes is most characteristic. For very small positive values of x, the value of y is very large, and as x approaches 0, y increases indefinitely. But the function is not defined for the value $x = 0$, for the product xy cannot equal 1 if x be zero. For numerically small but negative values of x, y is negative and numerically very large, and becomes numerically larger as x approaches 0. The locus thus approaches indefinitely near to the Y-axis, as x approaches zero.

Instead of saying that "y increases in value without limit," it is equally common to say "y becomes infinite;" in fact, "infinite" is merely the Latin equivalent of "no limit." It is often written $y = \infty$. This is a mere abbreviation for the longer expressions,

"y becomes infinite" or "y increases in value without limit." The student must be cautioned that the symbol ∞ does not stand for a *number*, and that "$y = \infty$" must not be interpreted in the same way that "$y = 5$" is interpreted.

As x increases from numerically large negative values to 0, y continually decreases and becomes negatively infinite (abbreviated $y = -\infty$). As x decreases from numerically large positive

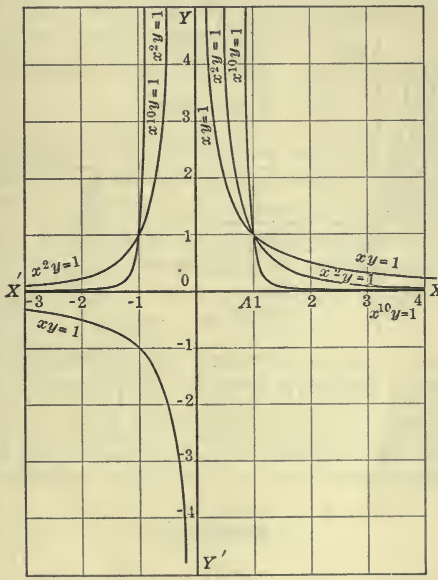


FIG. 34.—Hyperbolic Curves.

values to 0, y continually increases and becomes infinite. Thus, in the neighborhood of $x = 0$, y is **discontinuous,** and, in this case, the discontinuity is called an **infinite discontinuity.**

On account of the symmetry in $xy = 1$, if we look upon x as a function of y, all of the above statements may be repeated, merely interchanging x and y wherever they occur. Thus, there is an infinite discontinuity in x, as y passes through the value 0.

The lines XX' and YY' which these curves approach as near as we please, but never meet, are called the **asymptotes** of the hyperbola.

All other curves of the hyperbolic family, such as $yx^2 = 1$, $xy^2 = 1$, $y^3x^2 = 1$, $y^4x^4 = 1$ and the like, approach the X- and Y-axes as asymptotes. The rates at which they approach the axes depends upon the relative magnitudes of the exponents of the powers of x and y; the quadrants in which the branches lie depend upon the oddness or evenness of these exponents.

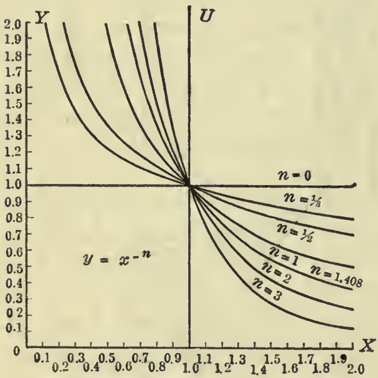


FIG. 35.—Hyperbolic Curves in the First Quadrant. $y = 1/x^{1.408}$ is the Adiabatic Curve for Air.

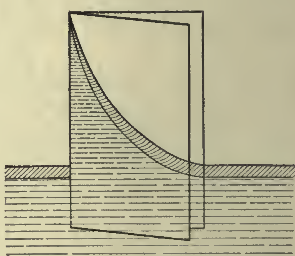


FIG. 36.—A Hyperbola Formed by Capillary Action of Two Converging Plane Plates.

Exercises

1. Draw accurately upon squared paper the loci, $xy = 1$, $xy^2 = 1$, $x^2y = 1$, $xy^3 = 1$.

2. Show that the curves of the hyperbolic type lie in the rectangular regions shown plain, or not cross-hatched, in Fig. 33.

3. In what quadrants do the branches of $x^{15}y^7 = 1$ lie?

4. How does the locus of $x^2y^2 = 1$ differ from that of $xy = 1$?

5. Sketch, showing the essential character of each locus, the curves $x^2y^3 = 1$, $x^{10}y = 1$, $x^{1000}y = 1$.

6. Show that $xy = a$ passes through the point $(\sqrt{a}, \sqrt{a})$; that $xy = a^2$ passes through (a, a) and can be made from $xy = 1$ by "stretching" (if $a > 1$) both abscissas and ordinates of $xy = 1$ in the ratio $1:a$.[1]

24. Curves Symmetrical to Each Other. Some of the facts of symmetry respecting two portions of the same parabola or hyperbola may be readily extended by the student to other curves. First answer the following questions:

How are the points (a, b) and $(-a, b)$ related to the Y-axis?

How are the points (a, b) and $(a, -b)$ related to the X-axis?

How are the points (a, b) and (b, a) related to the line $y = x$? Prove the result by plane geometry.

The following may then be readily proved by the student:

Theorems on Loci

I. *If x be replaced by $(-x)$ in any equation containing x and y, the new graph is the reflection of the former in the axis YY'.*

II. *If y be replaced by $(-y)$ in any equation containing x and y, the new graph is the reflection of the former in the axis XX'.*

III. *If x and y be interchanged in any equation containing x and y, the new graph is the reflection of the former one in the line $y = x$.*

25. The Variation of the Power Function. The symmetry of the graphs of the power function with respect to certain lines and points, while of interest geometrically, nevertheless does not constitute the most important fact in connection with these functions. Of more importance is *the law of change of value* or *the law by which the function varies.* Thus returning to a table of values for the power function x^2 for the first quadrant,

x	0	1/2	1	3/2	2	5/2	3
x^2	0	1/4	1	9/4	4	25/4	9

we note that as x changes from 0 to 1/2 the function grows by the small amount 1/4. As x changes from 1/2 by another increment of 1/2 to the value 1, the function increases by 3/4 to the value 1. As x grows by successive steps or increments of 1/2 unit each, it is seen that x^2 grows by increasingly greater and greater steps, until finally the change in x^2 produced by a small change in x

[1] To "elongate" or "stretch" in the ratio 2:3 means to change the length of a line segment so that (original length): (new or stretched length) = 2:3.

becomes very large. Thus the step by step increase in the function is a rapidly augmenting one. Even more rapidly does the function x^3 gain in value as x grows in value. On the contrary, for positive values of x the power functions $1/x$, $1/x^2$, $1/x^3$, etc., *decrease* in value as x grows in value. Referring to the definition of the slope of a curve given in **§14,** we see that the parabolic curves have a positive slope in the first quadrant, while the hyperbolic curves have always a negative slope in the first quadrant.

The law of the power function is stated in more definite terms in **§34.** That section may be read at once, and then studied a second time in connection with the practical work which precedes it.

26. Increasing and Decreasing Functions. As a point passes from left to right along the X-axis, x increases algebraically. As a point moves up on the Y-axis, y increases algebraically and as it moves down on the Y-axis, y decreases algebraically. An **increasing function of x** is one such that as x increases algebraically, y, or the function, also increases algebraically. By a **decreasing function of x** is meant one such that as x increases algebraically, y decreases algebraically. Graphically, an increasing function is indicated by a rising curve as a point moves along it from left to right. The power function $y = x^n$ (n positive) is an increasing function of x in the first quadrant. The power function $y = x^{-n}$ ($-n$ negative) in the first quadrant is a decreasing function of x.

The power function $y = x^3$ is an increasing function for all values of x while $y = x^2$ is a decreasing function in the second quadrant but an increasing function in the first quadrant. In a case like $y = \pm x^{3/2}$, where y has two values for each positive value of x, it is seen that one of these values increases with x while the other decreases with x.

Exercises

1. Consider the function $y = + x^{1/2}$ and construct its locus. As x grows by successive steps of one unit each, does the function grow by increasingly greater and greater steps or not? Why? Is the slope of the curve an increasing or a decreasing function of x?

2. Does the algebraic value of the slope of $xy = 1$ increase with x in the first quadrant?

3. As x changes from -5 to $+5$ does the slope of $y = x^2$ always increase algebraically?

4. Express in the language of mathematics the fact that the curves $y = x^n$, when n is a rational number greater than unity, are *concave upward.*

ANSWER: "When n is greater than unity, the slope of the curve increases as x increases."

Express in a similar way the fact that the curves $y = x^{1/n}$ are *concave downward.*

27. The Graph of the Power Function when x^n has a Coefficient. If numerical tables be prepared for the equations

$$y = x^2$$

and

$$y' = 3x^2$$

then for like values of x each ordinate, y', of the second curve will be three fold the corresponding ordinate, y, of the first curve. It is obvious that the curve

$$y' = ax^n \tag{1}$$

and the curve

$$y = x^n \tag{2}$$

are similarly related; the ordinate y' of any point of the first locus can be made from the corresponding ordinate y (*i.e.*, the ordinate having the same abscissa) of the second by multiplying the latter by a. If a be positive and greater than unity, this corresponds to *stretching* or elongating all ordinates of (2) in the ratio $1 : a$; if a be positive and less than unity, it corresponds to *contracting* or shortening all ordinates of (2) in the ratio $1 : a$.

For example, the graph of $y' = ax^n$ can be made from the graph of $y = x^n$ if the latter be first drawn upon sheet rubber, and if then the sheet be uniformly stretched in the y direction in the ratio $1 : a$. If the curve be drawn upon sheet rubber which is already under tension in the y direction and if the rubber be allowed to contract in the y direction, the resulting curve has the equation $y = ax^n$ *where a is a proper fraction or a positive number less than unity.*

The above results are best kept in mind when expressed in a slightly different from. The equation $y' = a \cdot x^n$ can, of course, be written in the from $(y'/a) = x^n$. Comparing this with the equation $y = x^n$, we note that $(y'/a) = y$ or $y' = ay$, therefore we may conclude generally that *substituting* (y'/a) *for* y *in the equation of any curve multiplies all of the ordinates of the curve by* a. For example, after substituting $(y'/2)$ for y in any equation, the new ordinate y' must be twice as large as the old ordinate y, in order that the equation remain true for the same value of x.

In the same manner changing the equation $y = x^n$ to $y = \left(\dfrac{x'}{a}\right)^n$, that is, *substituting* (x'/a) *for* x *in any equation multiplies all of the abscissas of the curve by* a. Multiplying all of the abscissas of a curve by a elongates or stretches all of the abscissas in the ratio[1] $1:a$ if $a > 1$, but contracts or shortens all of the abscissas if $a < 1$. As the above reasoning is true for the equation of any locus, we may state the results more generally as follows:

Theorems on Loci

IV. *Substituting* $\left(\dfrac{x}{a}\right)$ *for* x *in the equation of any locus multiplies all of the abscissas of the curve by* a.

V. *Substituting* $\left(\dfrac{y}{a}\right)$ *for* y *in the equation of any locus multiplies all of the ordinates of the curve by* a.

Note: It is not necessary to retain the symbols x' and y' to indicate new variables, if the change in the variable be otherwise understood.

Exercises

1. Without actual construction, compare the graphs $y = x^2$ and $y = 5x^2$; $y = x^2$ and $y = \dfrac{x^2}{2}$; $y = \dfrac{1}{x}$ and $y = \dfrac{2}{x}$; $y = x^3$ and $y = 2x^3$; $y = x^{3/2}$ and $y = \dfrac{x^{3/2}}{2}$.

2. Without actual construction, compare the graphs $y = x^2$ and $y = \left(\dfrac{x}{2}\right)^2$; $y = x^3$ and $\dfrac{y}{2} = x^3$; $y = x^3$ and $y = \left(\dfrac{x}{3}\right)^3$; $y = x^2$ and $\dfrac{y}{3} = x^2$.

[1] See footnote, p. 57.

3. Compare $y^2 = x^3$ and $y^2 = \left(\frac{x}{2}\right)^3$; $y^2 = x^3$ and $\left(\frac{y}{3}\right)^2 = x^3$; $y^2 = x^3$ and $\left(\frac{y}{3}\right)^2 = \left(\frac{x}{2}\right)^3$; $y = \frac{1}{x^2}$ and $\frac{y}{2} = \frac{1}{x^2}$.

28. Orthographic Projection. In elementary geometry we learned that the projection of a given point P upon a given line or plane is the foot of the perpendicular dropped from the given point upon the given line or plane. Likewise if perpendiculars be dropped from the end points A and B of any line segment AB upon a given line or plane, and if the feet of these perpendiculars be called P and Q, respectively, then the line segment PQ is called the **projection** of the line AB. Also, if perpendiculars be dropped from all points of a given curve AB upon a given plane MN, the

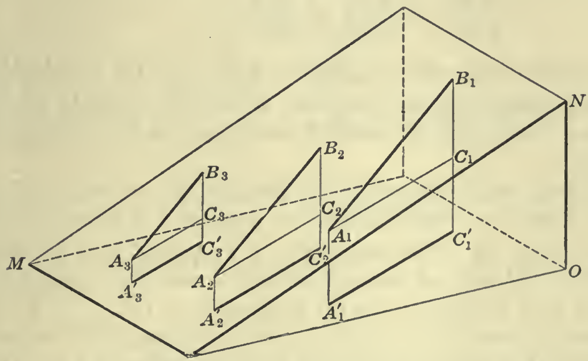


FIG. 37.—Orthographic Projection of Line Segments

locus of the feet of all of the perpendiculars so drawn is called the *projection of the given curve upon the plane MN*.

To emphasize the fact that the projections were made by using perpendiculars to the given plane, it is customary to speak of them as **orthogonal** or **orthographic** projections.

The shadow of a hoop upon the ground is not the orthographic projection of the hoop unless the rays of light from the sun strike *perpendicular* to the ground. This would only happen in our latitude upon a non-horizontal surface.

The shortening by a given fractional amount of all of a set of parallel line segments of a plane may be brought about geometrically by orthographic projection of all points of the line segments upon a second plane. For, in Fig. 37, let A_1B_1, A_2B_2, A_3B_3, etc., be parallel line segments lying in the plane MN. Let their projections on any other plane be $A'_1C'_1$, $A'_2C'_2$, $A'_3C'_3$, etc., respectively. Draw A_2C_2 parallel to $A'_2C'_2$ and A_1C_1 parallel to $A'_1C'_1$, etc. Then since the right triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, etc., are similar,

$$\frac{A_1B_1}{A_1C_1} = \frac{A_2B_2}{A_2C_2} = \frac{A_3B_3}{A_3C_3}$$

Call this ratio a. It is evident that $a > 1$. Substitute the equals: $A'_1C'_1 = A_1C_1$, $A'_2C'_2 = A_2C_2$, etc. Then:

$$\frac{A_1B_1}{A'_1C'_1} = \frac{A_2B_2}{A'_2C'_2} = \frac{A_3B_3}{A'_3C'_3} = \cdot \cdot \cdot = \frac{a}{1}$$

The numerators are the original line segments; the denominators are their projections on the plane MO. The equality of these fractions shows that the parallel lines have all been shortened in the ratio $a:1$.

The above work shows that to produce the curve $y = (x/a)^n$, $(a < 1)$, from $y = x^n$ by orthographic projection it is merely necessary to project all of the abscissas of $y = x^n$ upon a plane passing through YOY' making an angle with OX such that unity on OX projects into a length a on the projection of OX. To produce the curve $y = ax^n$ $(a < 1)$ from $y = x^n$ by orthographic projection it is merely necessary to project all of the ordinates of $y = x^n$ upon a plane passing through XOX' making an angle with OY such that unity on OY projects into the length a on the projection of OY.

To lengthen all ordinates of a given curve in a given ratio, $1 : a$, the process must be reversed; that is, erect perpendiculars to the plane of the given curve at all points of the curve, and cut them by a plane passing through XOX' making an angle with OY such that a length a $(a > 1)$ measured on the new Y-axis projects into unity on OY of the original plane.

29. Change of Unit. To produce the graph of $y = 10x^2$ from that of $y = x^2$, the stretching of the ordinates in the ratio $1:10$ need not actually be performed. If the unit of the vertical scale of $y = x^2$ be taken $1/10$ of that of the horizontal scale, and the proper numerical values be placed upon the divisions of the scales, then obviously the graph of $y = x^2$ may be used for the graph of $y = 10x^2$. Suitable change in the unit of measure on one or both of the scales of $y = x^n$ is often a very desirable method of representing the more general curve $y = ax^n$.

An interesting example is given in Fig. 38. The period of vibration of a simple pendulum is given by the formula $T = \pi\sqrt{l/g}$. When $g = 981$ cm. per second per second (abbreviated cm./sec.²) this gives $T = 0.1003\sqrt{l}$, which for many purposes is sufficiently accurate when written $T = 0.10\sqrt{l}$. In this equation T must be in seconds and l in centimeters. Thus when $l = 100$ cm., $T = 1$ sec., so that the graph may be made by drawing the parabola $y = \sqrt{x}$ from the pattern previously made and then attaching the proper numbers to the scales, as shown in Fig. 38.

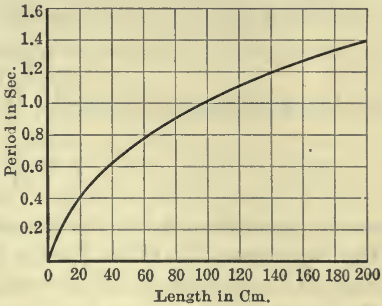


FIG. 38.—Relation of Length of a Simple Pendulum to Period of Vibration.

30. Variation. The relation between y and x expressed by the equation $y = ax^n$, where n is any positive number, is often expressed by the statement "y varies as the nth power of x," or by the statement "*y is proportional to x^n*." Likewise, the relation $y = a/x^n$, where n is positive, is expressed by the statement "y varies inversely as the nth power of x." The statement "the elongation of a coil spring is proportional to the weight of the suspended mass" tells us:

$$y = mx \qquad (1)$$

where y is the elongation (or increase in length from the natural or unloaded length) of the spring, and x is the weight suspended by the spring, *but it does not give us the value of m.* The value of m may readily be determined if the elongation corresponding to a given weight be given. Thus if a weight of 10 pounds when suspended from the spring produces an elongation of 2 inches in the length of the coil, then, substituting $x = 10$ and $y = 2$ in (1),

$$2 = m10$$

and hence

$$m = 1/5$$

If this spring be used in the construction of a spring balance, the length of a division of the uniform scale corresponding to 1 pound will be 1/5 inch.

A special symbol, $\propto$, is often used to express variation. Thus

$$y \propto 1/d^2$$

states that y varies inversely as d^2. It is equally well expressed by:

$$y = \frac{k}{d^2}$$

where k is a constant called the **proportionality factor.**

The statements "y varies jointly as u and v," and "y varies directly as u and inversely as v," mean, respectively:

$$y = auv$$

$$y = \frac{au}{v}$$

Thus the area of a rectangle varies jointly as its length and breadth, or,

$$A = kLB$$

If the length and breadth are measured in feet and A in square feet, k is unity. But, if L and B are measured in feet and A in acres, then $k = 1/43560$. If L and B are measured in rods and A in acres, then $k = 1/160$.

From Ohm's law, we say that the electric current in a circuit varies directly as the electromotive force and inversely as the resistance, or:

$$C \propto E/R \text{ or } C = kE/R$$

The constant multiplier is unity if C be measured in amperes, E in volts, and R in ohms, so that for these units

$$C = E/R$$

31. Illustrations from Science. Some of the most important laws of natural science are expressed by means of the power function[1] or graphically by means of loci of the parabolic or hyperbolic type.

The linear equation $y = mx$ is, of course, the simplest case of the power function and its graph, the straight line, may be regarded as the simplest of the curves of the parabolic type. The following illustrations will make clear the importance of the power function in expressing numerous laws of natural phenomena. Later the student will learn of two additional types of fundamental laws of science expressible by two functions entirely different from the power function now being discussed.

The instructor will ask oral questions concerning each of the following illustrations. The student should have in mind the general form of the graph in each case, but should remember that the *law of variation*, or the law of change of value which the functional relation expresses, is the matter of fundamental importance. The graph is useful primarily because it aids to form a mental picture of the law of variation of the function. The practical graphing of the concrete illustrations given below will not be done at present, but will be taken up later in §33.

(*a*) The pressure of a fluid in a vessel may be expressed in either pounds per square inch or in terms of the height of a column of mercury possessing the same static pressure. Thus we may write:

$$p = 0.492h \tag{1}$$

in which p is pressure in pounds per square inch and h is the height of the column of mercury in inches. The graph is the straight line through the origin of slope 492/1000. The constant 0.492 can be computed from the data that the weight of mercury is 13.6 times that of an equal volume of water and that 1 cubic foot of water weighs 62.5 pounds.

In this and the following equations, it must be remembered that *each letter represents a number*, and that no equation can be used until all the magnitudes involved are expressed in terms of the particular units which are specified in connection with that equation.

[1] For brevity ax^n as well as x^n will frequently be called a power function of x.

5

(*b*) The velocity of a falling body which has fallen from a state of rest during the time t, is given by

$$v = 32.2t \tag{2}$$

in which t is the time in seconds and v is the velocity in feet per second. If t is measured in seconds and v is in centimeters per second, the equation becomes[1] $v = 981t$. In either case the graph is a straight line, but the lines have different slopes.

(*c*) The space traversed by a falling body is given by

$$s = \tfrac{1}{2}gt^2 \tag{3}$$

or, in English units (s in feet and t in seconds):

$$s = 16.1t^2 \tag{4}$$

(*d*) The velocity of the falling body, from the height h is:

$$v = \sqrt{2gh} = \sqrt{64.4h} \tag{5}$$

The resistance of the air is not taken into account in formulas (2) to (5).

The formula equivalent to (5):

$$\tfrac{1}{2}mv^2 = mgh \tag{6}$$

where m is the mass of the body, expresses the equivalence of $\frac{1}{2}mv^2$, the kinetic energy of the body, and mgh, the work done by the force of gravity mg, working through the distance h.

[1] A full discussion of the process of changing formulas like the ones in the present section into a new set of units should be sought in text-books on physics and mechanics. The following method is sufficient for elementary purposes. First, write (for the present example) the formula $v = 32.2\ t$ where v is in ft./sec. and t is in seconds. For *any units of measure that may be used*, there holds a general relation $v = ct$, where c is a constant. To determine what we may call the *dimensions* of c, substitute for all letters in the formula the names of the units in which they are expressed, treating the names as though they were algebraic numbers. From $v = ct$ write, ft./sec. = c sec. Hence (solving for *dimensions* of c), c has dimensions ft./sec.^{2} Therefore in the given case, we know $c = 32.2$ ft./sec^2. To change to any other units simply substitute equals for equals. Thus 1 ft. = 30.5 cm., hence $c = 32.2 \times 30.5$ cm./sec.^{2} = 981 cm./sec.^{2}

To change velocity from mi./hr. to ft./sec. in formula (19) below, we have $R = 0.003\ V^2$ where R is in lb./sq. ft. and V is in mi./hr. Write the general formula $R = cV^2$. The dimensions of c are (lb. / ft.^{2}) $\div$ (mi.^{2}/hr.^{2}) or (lb. / ft.^{2}) $\times$ (hr.^{2}/mi.^{2}). In the given case we have the *value* of $c = 0.003$ (lb./ft.^{2}) $\times$ (hr.^{2}/mi^2). To change V to ft./sec., substitute equals for equals, namely 1 hr. = 3600 sec., 1 mi. = 5280 ft., or merely (approximately) mi./hr. = $\frac{3}{2}$ ft./sec.

(*e*) The intensity of the attraction exerted on a unit mass by the sun or by any planet varies inversely as the square of the distance from the center of mass of the attracting body. If r stand for that distance and if f be the force exerted on unit mass of the attracted body, then

$$f = \frac{m}{r^2} \tag{7}$$

The constant m is the value of the force when r is unity.

(*f*) The formula for the horse power transmissible by cold-rolled shafting is:

$$H = \frac{d^3N}{50} \tag{8}$$

where H is the horse power transmitted, d the diameter of the shaft in inches, and N the number of revolutions per minute.

The rapid variation of this function (as the cube of the diameter) accounts for some interesting facts. Thus doubling the size of the shaft operating at a given speed increases 8-fold the amount of power that can be transmitted, while the weight of the shaft is increased but 4-fold.

If H be constant, N varies inversely as d^3. Thus an old-fashioned 50-h.p. overshot water-wheel making three revolutions per minute requires about a 9-inch shaft, while a DeLaval 50-h.p. steam turbine making 16,000 revolutions per minute requires a turbine shaft but little over 1/2 inch in diameter.

(*g*) The period of the simple pendulum is

$$T = \pi\sqrt{l/g} \tag{9}$$

where T is the time of one swing in seconds, l the length of the pendulum in feet and g = 32.2 ft./sec.,^{2} approximately.

(*h*) The centripetal force on a particle of weight W pounds, rotating in a circle of radius R feet, at the rate of N revolutions per second is

$$F = \frac{4^2\pi WRN^2}{g} \tag{10}$$

or, if $g = 32.16$ ft./sec.^{2},

$$F = 1.2276WRN^2 \tag{11}$$

where F is measured in pounds. If N be the number of revolutions per minute, then

$$F = \frac{4\pi^2 WRN^2}{3600g} \tag{12}$$

$$= 0.000341 WRN^2 \tag{13}$$

(*i*) An approximate formula for the indicated horse power required for a steamboat is:

$$I.H.P. = \frac{S^3 D^{2/3}}{C} \tag{14}$$

where S is speed in knots, D is displacement in tons, and C is a constant appropriate to the size and model of the ship to which it is applied. The constant ranges in value from about 240, for finely shaped boats, to 200, for fairly shaped boats.

(*j*) Boyle's law for the expansion of a gas maintained at constant temperature is

$$pv = C \tag{15}$$

where p is the pressure and v the volume of the gas, and C is a constant. Since the density of a gas is inversely proportional to its volume, the above equation may be written in the form

$$p = c\rho \tag{16}$$

in which ρ is the density of the gas.

(*k*) The flow of water over a trapezoidal weir is given by

$$q = 3.37Lh^{3/2} \tag{17}$$

where q is the quantity in cubic feet per second, L is the length of the weir[1] in feet and h is the head of water on the weir, in feet.

(*l*) The physical law holding for the adiabatic expansion of air, that is, the law of expansion holding when the change of volume is not accompanied by a gain or loss of heat,[2] is expressed by

$$p = c\rho^{1.408} \tag{18}$$

[1] The instructor is expected fully to explain the meaning of the technical terms here used.

[2] Note that when a vessel containing a gas is insulated by a non-conductor of heat, so that no heat can enter or escape from the vessel, that the *temperature* of the gas will rise when it is compressed, or fall when it is expanded. Adiabatic expansion may be thought of, therefore, as taking place in an *insulated* vessel.

This is a good illustration of a power function with fractional exponent. The graph is not greatly different from the semi-cubical parabola

$$y = cx^{3/2}$$

(*m*) The pressure or resistance of the air upon a flat surface perpendicular to the current is given by the formula

$$R = 0.003V^2 \tag{19}$$

in which V is the velocity of the air in miles per hour and R is the resulting pressure upon the surface in pounds per square foot. According to this law, a 20-mile wind would cause a pressure of about 1.2 pounds per square foot upon the flat surface of a building. One foot per second is equivalent to about 2/3 mile per hour, so that the formula when the velocity is given in feet per second becomes:

$$R = 0.0013V^2 \tag{20}$$

(*n*) The power used to drive an aeroplane may be divided into two portions. One portion is utilized in overcoming the resistance of the air to the onward motion. The other part is used to sustain the aeroplane against the force of gravity. The first portion does "useless" work—work that should be made as small as possible by the shapes and sizes of the various parts of the machine. The second part of the power is used to form continuously anew the wave of compressed air upon which the aeroplane rides. Calling the total power[1] P, the power required to overcome the resistance P_r, and that used to sustain the aeroplane P_s, we have

$$P = P_r + P_s \tag{21}$$

We learn from the theory of the aeroplane that P_r varies as the cube of the velocity, while P_s varies inversely as V, so that

$$P_r = cV^3 \tag{22}$$

and

$$P_s = \frac{k}{V} \tag{23}$$

Thus at high velocity less and less power is required to sustain the aeroplane but more and more is required to meet the frictional

[1] Power (= work done per unit time) is measured by the unit *horse power*, which is 550 foot-pounds per second.

resistance of the medium. The law expressed by (23) that less and less power is required to sustain the aeroplane as the speed is increased is known as Langley's Law. From this law Langley was convinced that artificial flight was possible, for the whole matter seemed to depend primarily upon getting up sufficient speed. It is really this law that makes the aeroplane possible. An analogous case is the well-known fact that the faster a person skates, the thinner the ice necessary to sustain the skater. In this case

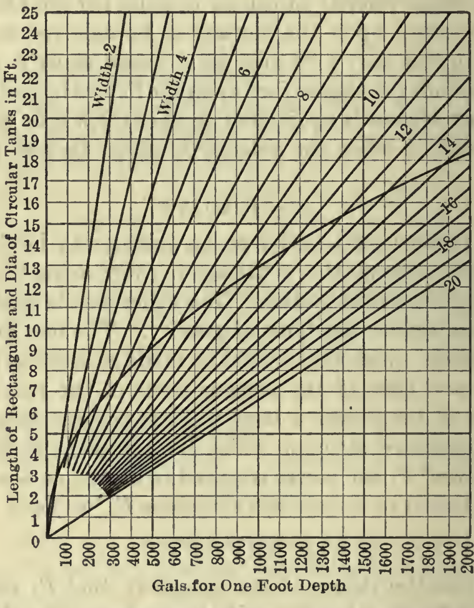


FIG. 39.—Capacity of Rectangular and Circular Tanks per Foot of Depth.

part of the energy of the skater is continually forming anew on the thin ice the wave of depression which sustains the skater, while the other part overcomes the frictional resistance of the skates on the ice and the resistance of the air.

(*o*) The capacity of cast-iron pipe to transmit water is often given by the formula:

$$q^{1.86} = 1.68hd^{3.25} \tag{24}$$

in which q is the quantity of water discharged in cubic feet per second, d is the diameter of the pipe in feet and h is the loss of head measured in feet of water per 1000 linear feet of pipe. This is a good illustration of the equation of a parabolic curve with complicated fractional exponents. The curve is very roughly approximate to the locus of the equation

$$y = c\sqrt{h}\,x^{3/2} \tag{25}$$

(p) The contents in gallons of a rectangular tank per foot of depth, b feet wide and l feet long, is

$$q = 7.5bl \tag{26}$$

The contents in gallons per foot of depth of a cylindrical tank d feet in diameter is

$$q = 7.5\,\pi d^2/4 \tag{27}$$

Fig. 39 shows the graph of (26) for various values of b and also shows to the same scale the graph of (27).

32. Rational and Empirical Formulas. A number of the formulas given above are capable of demonstration by means of theoretical considerations only. Such for example are equations (1), (2), (3), (4), (5), (7), (8), (9), (10), etc., although the constant coefficients in many of these cases were experimentally determined. Formulas of this kind are known in mathematics as **rational formulas.** On the other hand certain of the above formulas, especially equations (14), (17), (19), (22), (23), (24), including not only the constant coefficients but also the *law of variation of the function itself*, are known to be true only as the result of experiment. Such equations are called **empirical formulas.** Such formulas arise in the attempt to express by an equation the results of a series of laboratory measurements.

For example, the density of water (that is, the mass per cubic centimeter or the weight per cubic foot) varies with the temperature of the water. A large number of experimenters have prepared accurate tables of the density of water for wide ranges of temperature centigrade, and a number of very accurate empirical formulas have been ingeniously devised to express the results, of which the following four equations are samples:

Empirical formulas *for the density, d, of water in terms of temperature centigrade, θ.*

$$(a)\quad d = 1 - \frac{96(\theta - 4)^2}{10^7}$$

$$(b)\quad d = 1 - \frac{93(\theta - 4)^{1.982}}{10^7}$$

$$(c)\quad d = 1 - \frac{6\theta^2 - 36\theta + 47}{10^6}$$

$$(d)\quad d = 1 + \frac{0.485\theta^3 - 81.3\theta^2 + 602\theta - 1118}{10^7}$$

Exercises

1. Among the power functions named in the above illustrations, pick out examples of increasing functions and of decreasing functions.

2. Under the same difference of head or pressure, show by formula (24) that an 8-inch pipe will transmit much more than double the quantity of water per second that can be transmitted by a 4-inch pipe.

3. Wind velocities during exceptionally heavy hurricanes on the Atlantic coast are sometimes over 140 miles per hour. Show that the wind pressure on a flat surface during such a storm is about fifty times the amount experienced during a 20-mile wind.

4. Show that for wind velocities of 10, 20, 40, 80, 160 miles per hour (varying in geometrical progression with ratio 2), the pressure exerted on a flat surface is 0.3, 1.2, 4.8, 19.2, 76.8 pounds per square foot respectively (varying in geometrical progression with ratio 4).

5. A 300-h.p. DeLaval turbine makes 10,000 revolutions per minute. Find the necessary diameter of the propeller shaft.

6. A railroad switch target bent over by the wind during a tornado in Minnesota indicated an air pressure due to a wind of 600 miles per hour. Show that the equivalent pressure on a flat surface would be 7.5 pounds per square inch.

7. Show that a parachute 50 feet in diameter and weighing 50 pounds will sustain a man weighing 205 pounds when falling at the rate of 10 feet per second.

Suggestion: Use approximate value $\pi = 22/7$ in finding area of parachute from formula for circle, πr^2, and use formula (20) above.

8. Show that empirical formulas (*a*) and (*b*) for the density of water reduce to a power function if the origin be taken at $\theta = 4, d = 1$.

33. Practical Graphs of Power Functions. The graphs of the power function

$$y = x^2, \quad y = x^3, \quad y = 1/x, \quad y = x^{3/2}, \quad \text{etc.,} \tag{1}$$

can, of course, be made the basis of the laws concretely expressed by equations (1) to (27) of **§31**. If, however, the graph of a scientific formula is to serve as a numerical table of the function for actual use in practical work, then there is much more labor in the proper construction of the graph than the mere plotting of the abstract mathematical function. The size of the unit to be selected, the range over which the graph should extend, the permissible course of the curve, become matters of practical importance.

If the apparent slope[1] of a graph departs too widely from $+1$ or -1, it is desirable to make an abrupt change of unit in the vertical or the horizontal scale, so as to bring the curve back to a desirable course, for it is obvious that numerical readings can best be taken from a curve when it crosses the rulings of the coördinate paper at apparent slopes differing but little from ± 1.

The above suggestions in practical graphing are illustrated by the following examples:

Graph the formula (equation (8), **§31**), for the horse power transmissible by cold-rolled shafting

$$H = \frac{d^3N}{50} \tag{2}$$

in which d is the diameter in inches and N is the number of revolutions per minute. The formula is of interest only for the range of d between 0 and 24 inches, as the dimensions of ordinary shafting lie well within these limits. Likewise one would not ordinarily be interested in values of N except those lying between 10 and 3000 revolutions per minute. Fig. 40 shows a suitable graph of this formula for the range $1 < d < 10$ for the fixed value of $N = 100$. In order properly to graph this function, three different scales have been used for the ordinate H, so that the slope of the curve may not depart too widely from unity.

[1] Of course the *real* slope of a curve is independent of the scales used. By *apparent* slope = 1 is meant that the graph appears to cut the ruling of the squared paper at about 45°.

If similar graphs be drawn for $N = 200$, $N = 300$, $N = 400$, etc., a set of parabolas is obtained from which the horse power of shafting for various speeds of rotation as well as for various diameters may be obtained at once. A set of curves systematically constructed in a manner similar to that just described, is often called a **family of curves.** Fig. 39 shows a family of straight lines expressing the capacity of rectangular tanks corresponding to the various widths of the tanks.

Inasmuch as many of the formulas of science are used only for positive values of the variables, it is only necessary in these cases to graph the function in the first of the four quadrants. For such problems the origin may be taken at the lower left corner of the coördinate paper so that the entire sheet becomes available for the curve in the first quadrant.

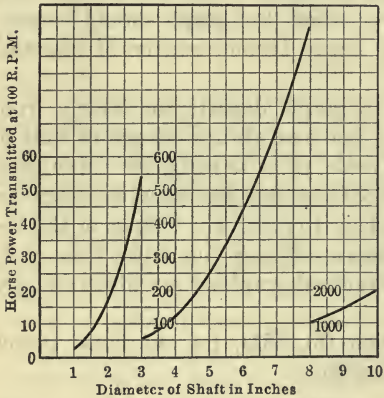


Fig. 40.—Capacity at 100 R.P.M. of Cold-rolled Shafting to Transmit Power.

The above illustrations are sufficient to make clear the importance in science of the functions now being discussed. The following exercises give further practice in the useful application of the properties of the functions.

Exercises

The graphs for the following problems are to be constructed upon rectangular coördinate paper. The instructions are for centimeter paper (form $M1$) ruled into 20×25 cm. squares. In each case the units for abscissa and for ordinates are to be so selected as best to exhibit the functions, considering both the workable range of values of the variables and the suitable slope of the curves.

The student should read **§12** a second time before proceeding with the following exercises, giving especial care to instructions (4), (5) and (6) given in that section.

1. Classify the graphs of formulas (1) to (27), **§31**, as to parabolic or hyperbolic type.

2. Graph the formula $v^2 = 2gh$, or $v = \sqrt{2gh} = 8.02h^{1/2}$, if h range between 1 and 100, the second and foot being the units of measure. See formula (5), **§31.**

The following table of values is readily obtained:

h	1	5	10	20	30	40	50	60	70	80	90	100
v	8.02	17.9	25.3	35.8	43.9	50.7	56.7	62.1	67.1	71.7	76.0	80.2

Use 2 cm. = 10 feet as the horizontal unit for h, and 2 cm. = 10 feet per second as the vertical unit for v. The graph is then readily constructed without change of unit or other special expedient.

3. Graph the formula $q = 3.37Lh^{3/2}$ for $L = 1$, for $h = 0$, 0.1, 0.2, 0.3, 0.4, 0.5. See formula (17), **§31.** Use 4 cm. = 0.1 for horizontal unit for h and 2 cm. = 0.1 for vertical unit for q.

4. Draw a curve showing the indicated horse power of a ship $I.H.P. = S^3D^{2/3}/C$ for $C = 200$ if the displacement $D = 8000$ tons, and for the range of speeds $S = 10$ to $S = 20$ knots. See formula (14), **§31.**

For the vertical unit use 1 cm. = 1000 h.p. and for the horizontal unit use 2 cm. = 1 knot. Call the lower left-hand corner of the paper the point ($S = 10$, $I.H.P. = 0$).

5. From the formula expressing the centripetal force in pounds of a rotating body,

$$F = 0.000341WRN^2$$

draw a curve showing the total centripetal force sustained by a 36-inch automobile tire weighing 25 pounds, for all speeds from 10 to 40 miles per hour. See formula (13), **§31.**

Miles per hour must first be converted into revolutions per minute by dividing 5280 by the circumference of the tire and then dividing the result by 60. This gives

$$1 \text{ mile an hour} = 9\tfrac{1}{3} \text{ revolutions a minute}$$

If V be the speed in miles per hour the formula for F becomes

$$F = 0.000341(1.5)25(9\tfrac{1}{3})^2V^2 = 1.11V^2$$

For horizontal scale let 4 cm. = 10 miles an hour and for the vertical scale let 1 cm. = 100 pounds.

6. Draw a curve from the formula $f = m/r^2$ showing the acceleration of gravity due to the earth at all points between the surface of the earth and a point 240,000 miles (the distance to the moon) from the center, if $f = 32.2$ when radius of the earth $= 4000$ miles.

It is convenient in constructing this graph to take the radius of the earth as unity, so that the graph will then be required of $f = 32.2/r^2$ from $r = 1$ to $r = 60$. In order to construct a suitable curve several changes of units are desirable. See Fig. 41. One centimeter represents one radius (4000 miles) from $r = 0$ to $r = 10$, after which the scale is reduced to 1 cm. $= 10r$. In the vertical direction the scale is 4 cm. $=$ 10 feet per second for $0 < r < 5$, 4 cm. $=$ 1 foot a second for $5 < r < 10$, and 4 cm. $=$ 0.1 foot a second for

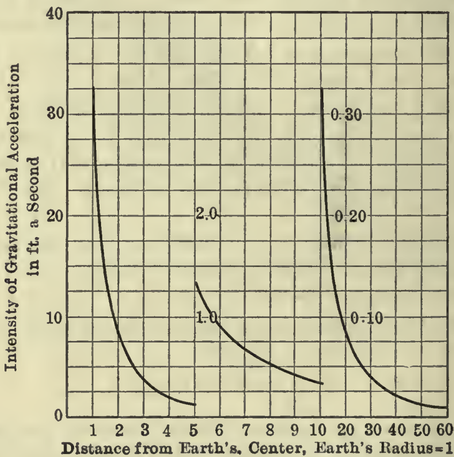


FIG. 41.—Gravitational Acceleration at Various Distances from the Earth's Center. The moon is distant approximately 60 earth's radii from the center of the earth.

$10 < r < 60$. Even with these four changes of units just used the first and third curves are somewhat steep. The student can readily improve on the scheme of Fig. 41 by a better selection of units.

34. The Law of the Power Functions. Sufficient illustrations have been given to show the fundamental character of the power function as an expression of numerous laws of natural phenomena. How may a functional dependence of this sort be expressed in words? If a series of measurements are made in the laboratory, so as to produce a numerical table of data covering certain phenomena, how can it be determined whether or not a power function can be written down which will express the law (that is, the function) defined by the numerical table of laboratory results? The answers to these questions are readily given. Consider first the law of the falling body

$$s = 16.1t^2 \tag{1}$$

Make a table of values for values of t = 1, 2, 4, 8, 16 seconds, as follows:

t	1	2	4	8	16
s	16.1	64.4	257.6	1030.4	4121.6

The values of t have been so selected that t increases by a fixed multiple; that is, each value of t in the sequence is twice the preceding value. From the corresponding values of s it is observed that s also increases by a fixed multiple, namely 4.

Similar conclusions obviously hold for any power function. Take the general case

$$y = ax^n \tag{2}$$

where n is any exponent, positive, negative, integral or fractional. Let x change from any value x_1 to a multiple value mx_1 and call the corresponding values of y, y_1 and y_2. Then we have

$$y_1 = ax_1^n \tag{3}$$

and

$$y_2 = a(mx_1)^n = am^nx_1^n \tag{4}$$

Divide the members of (4) by the members of (3) and we have

$$\frac{y_2}{y_1} = m^n \tag{5}$$

That is, if x in any power function change by the fixed multiple m, then the value of y will change by a fixed multiple m^n. Thus the law of the power function may be stated in words in either of the two following forms:

In any power function, if x change by a fixed multiple, y will change by a fixed multiple also.

In any power function, if the variable increase by a fixed percent, the function will increase by a fixed percent also.

This test may readily be applied to laboratory data to determine whether or not a power function can be set up to represent as a formula the data in hand. To apply this test, select at several places in one column of the laboratory data, pairs of numbers which change by a selected fixed percent, say 10 percent, or 20 percent, or any convenient percent. Then the corresponding pairs of numbers in the other column of the table must also be related by a fixed percent (of course, not in general the same as the first-named percent), provided the functional relation is expressible by means of a power function. If this test does not succeed, then the function in hand is not a power function.

Since the fixed percent for the function is m^n if the fixed percent for the variable be m, the possibility of determining n exists, since the table of laboratory data must yield the numerical values of both m and m^n.

35. Simple Modifications of the Parabolic and of the Hyperbolic Types of Curves. In the study of the motion of objects it is convenient to divide bodies into two classes: first, bodies which retain their size and shape unaltered during the motion; second, bodies which suffer change of size or shape or both during the motion. The first class of bodies are called **rigid bodies**; a moving stone, the reciprocating or rotating parts of a machine, are illustrations. The second class of bodies are called **elastic bodies**; a piece of rubber during stretching, a spring during elongation or contraction, a rope or wire while being coiled, the water flowing in a set of pipes, are all illustrations of this class of bodies.

When a body changes size or shape the motion is called a **strain.**

Bodies that preserve their size and shape unchanged may possess motion of two simple types: (1) **Rotation,** in which all particles of the body move in circles whose centers lie in a straight line called the **axis of rotation,** which line is perpendicular to the plane of the circles, and (2) **translation,** in which each straight line of the body remains fixed in direction.

We have already noted that the curve

$$\frac{y_1}{a} = x^n \tag{1}$$

can be made from the curve

$$y = x^n \tag{2}$$

by multiplying all the ordinates of (2) by a. The effect is either to elongate or to contract all of the ordinates, depending upon whether $a > 1$ or $a < 1$ respectively. The substitution of (y_1/a) for y has therefore produced a motion or strain in the curve $y = x^n$, which in this case is the object whose motion is being studied. Likewise

$$y = (x_1/a)^n \tag{3}$$

can be made from

$$y = x^n \tag{4}$$

by multiplying all of the abscissas of (4) by a. The effect is either to stretch or to contract all of the abscissas, depending upon whether $a > 1$, or $a < 1$ respectively.

In general, if a curve have the equation

$$y = f(x) \tag{5}$$

then

$$y = f(x_1/a) \tag{6}$$

i made from curve (5) by lengthening or stretching the XY-plane uniformly in the x direction in the ratio $1 : a$.

The statement just given is made on the assumption that $a > 1$. If $a < 1$ then the above statements must be changed by substituting *shorten* or *contract* for *elongate* or *stretch*.

The reasons for the above conclusions have been previously stated: substituting $\left(\frac{x_1}{a}\right)$ everywhere as the equal of x multiplies all of the abscissas by a. That is, if $\left(\frac{x_1}{a}\right) = x$, then $x_1 = ax$, so that x_1 is a-fold the old x.

We shall now explain how certain other of the motions mentioned above may be given to a locus by suitable substitution for x and y.

36. Translation of Any Locus. If a table of values be prepared for each of the loci

$$y = x^2 \tag{1}$$

$$y = (x_1 - 3)^2 \tag{2}$$

as follows:

x	-2	-1	0	1	2	3	4
y	4	1	0	1	4	9	16

x_1	-2	-1	0	1	2	3	4	5	6
y	25	16	9	4	1	0	1	4	9

and then if the graph of each be drawn, it will be seen that the curves differ only in their location and not at all in shape or size. The reason for this is obvious: if $(x_1 - 3)$ be substituted for x in any equation, then since $(x_1 - 3)$ has been put equal to x, it follows that $x_1 = x + 3$, or the new x, namely x_1, is greater than the original x by the amount 3. This means that the new longitude of each point of the locus after the substitution is greater than the old longitude by the fixed amount 3. Therefore the new locus is the same as the original locus *translated to the right the distance* 3.

The same reasoning applies if $(x_1 - a)$ be substituted for x, and the amount of translation in this case is a. The same reasoning applies also to the general case $y = f(x)$ and $y = f(x_1 - a)$, the latter curve being the same as the former, translated the distance a in the x direction.

As it is always easy to distinguish from the context the new x from the old x, it is not necessary to use the symbol x_1, since the old and new abscissas may both be represented by x. The following theorems may then be stated:

Theorems on Loci

VI. *If $(x - a)$ be substituted for x throughout any equation, the locus is translated a distance a in the x direction.*

VII. *If $(y - b)$ be substituted for y in any equation, the locus is translated the distance b in the y direction.*

These statements are perfectly general: if the signs of a and b are negative, so that the substitutions for x and y are of the form $x + a'$ and $y + b'$, respectively, then the translations are to the *left* and *down* instead of to the *right* and *up*.

Sometimes the motion of translation may seem to be disguised by the position of the terms a or b. Thus the locus $y = 3x + 5$

is the same as the locus $y = 3x$ translated upward the distance 5, for the first equation is really $y - 5 = 3x$, from which the conclusion is obvious.

Exercises

1. Compare the curves: (1) $y = 2x$ and $y = 2(x - 1)$; (2) $y = x^3$ and $y = (x - 4)^3$; (3) $y = x^3$ and $y - 3 = x^3$; (4) $y = x^{3/2}$ and $y = (x - 5)^{3/2}$; (5) $y = 5x^2$ and $y = 5(x + 3)^2$; (6) $y = 2x^3$ and $y = 2(x - k)^3$; (7) $y = 2x^3$ and $y = 2x^3 + k$; (8) $y + 7 = x^2$ and $y = x^2$ and $y - 7 = x^2$; (9) $3y^2 = 5x^3$ and $3(y - b)^2 = 5(x - a)^3$.

2. Compare the curves: (1) $y = x^3$ and $y = (x/2)^3$; (2) $y = x^3$ and $y = x^3/8$; (3) $y = x^3$ and $y/2 = x^3$; (4) $y = x^3$ and $y = 2x^3$; (5) $y^2 = 3x^3$ and $(y/5)^2 = 3(x/7)^3$; (6) $y^2 = x^3$ and $y^2 = (3x)^3$; (7) $y = x^2$ and $y = 4x^2$ (note: explain in two ways); (8) $y = x^3$ and $2y = x^3$ and $y = 27x^3$.

3. Translate the locus $y = 2x^3$; (1) 3 units to the right; (2) 4 units down; (3) 5 units to the left.

4. Elongate three-fold in the x direction the loci: (1) $y^2 = x$; (2) $3y = x^3$; (3) $y^2 = 2x^3$; (4) $y = 2x + 7$.

5. Multiply by 1/2 the ordinates of the loci named in exercise 4.

6. Show that $y = \dfrac{1}{x + b}$ and $y = \dfrac{1}{x - b}$ are hyperbolas.

7. Show that $y = \dfrac{x}{x + b}$ is a hyperbola.

NOTE: Divide the numerator by the denominator, obtaining the equation $y = 1 - \dfrac{b}{x + b}$.

8. Show that $y = \dfrac{x + a}{x + b}$ is a hyperbola, namely, the curve $xy = a - b$ translated to a new position.

37. Shearing Motion. An important strain of the XY-plane occurs if we derive

$$y = f(x) + mx \tag{1}$$

from

$$y' = f(x) \tag{2}$$

Graphically, the curve (1) is seen to be formed by the addition of the ordinates of the straight line $y'' = mx$ to the corresponding ordinates of $y' = f(x)$. Thus, in Fig. 42, the graph of the function $x^3 + x$ is made by adding the corresponding ordinates of $y' = x^3$ and $y'' = x$. Mechanically, this might be done by drawing the curve on the edge of a pack of cards, and then slipping the cards over each other uniform amounts. The change of the shape of a body, or the strain of a body, here illustrated, is called **lamellar motion** or **shearing motion.** It is a form of motion of very great importance.

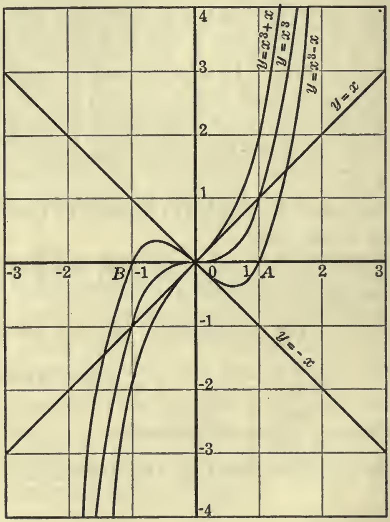


FIG. 42.—The Shear of the Cubical Parabola $y = x^3$ in the line $y = x$, and also in the Line $y = -x$.

We shall speak of the locus $y = f(x) + mx$ as *the shear of the curve* $y = f(x)$ *in the line* $y = mx$.

THEOREMS ON LOCI

VIII. *The addition of the term mx to the right side of* $y = f(x)$ *shears the locus* $y = f(x)$ *in the line* $y = mx$.

The locus

$$y = ax^3 + mx + b$$

is made from $y = x^3$ by a combination of (1) a uniform elongation a], (2) a shearing motion [m], and (3) a translation [b]. Either motion may be changed in sense by changing the sign of a, m, or b, respectively.

The student may easily show that the effect of a shearing motion upon the straight line $y = mx + b$ is merely a rotation about the fixed point $(0, b)$. The line is really *stretched* in the direction

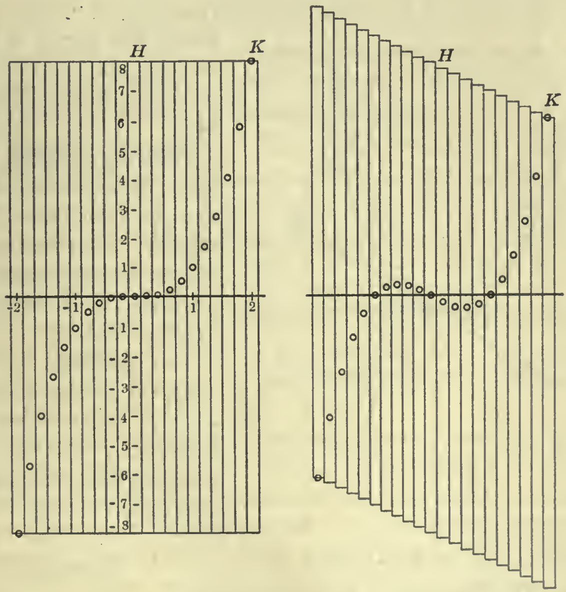


FIG. 43.—Shearing Motion Illustrated by the Slipping of the Members of Pack of Cards.

of its own length, but this does not change the shape of the line nor does it change the line geometrically. A *line segment* (that is, a line of finite length) would be modified, however.

The parabola $y = x^2$ is transformed under a shearing motion in a most interesting way. For, after shear, $y = x^2$ becomes:

$$y = x^2 + 2mx \qquad (3)$$

where, for convenience, the amount of the shearing motion is represented by $2m$ instead of by m. Writing this in the form

$$y = x^2 + 2mx + m^2 - m^2$$

or,

$$y = (x + m)^2 - m^2$$

$$y + m^2 = (x + m)^2 \tag{4}$$

we see that (4) can be made from the parabola $y = x^2$ by translating the curve *to the left* the amount m and *down* the amount m^2. (See Fig. 44.)

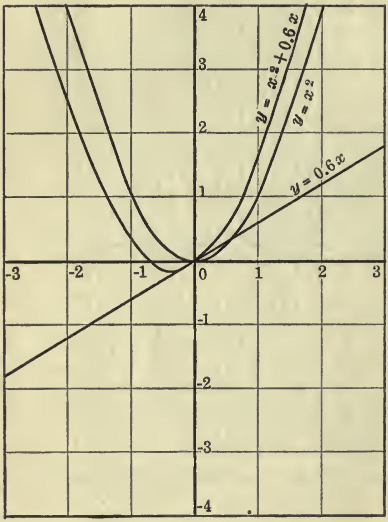


FIG. 44.—The Shear of $y = x^2$ in the line $y = 0.6x$.

Shearing motion, therefore, rotates the straight line and translates the parabola. The effect on other curves is much more complicated, as is seen from Figs. 42 and 43.

The parabola $y = x^2$ is identical in size and shape with $y = x^2 + mx + b$. Likewise, $y = ax^2 + bx + c$ is a parabola differing only in position from $y = ax^2$.

Exercises

1 Explain how the curve $y = x^3 + 2x$ may be made from the curve $y = x^3$. How can the curve $y = 2x^3 + 3x$ be made from the curve $y = 2x^3$?

2. Find the coördinates of the lowest point of $y = x^2 - 4x$; that is, put this equation in the form $y - b = (x - a)^2$.

3. Compare the curves $y = x^3 + 2x$ and $y = x^3 - 2x$. (Do not *draw* the curves.)

4. Explain the curve $y = 1/x + 2x$ from a knowledge of $y = 1/x$ and of $y = 2x$.

38. Rotation of a Locus. The only simple type of displacement of a locus not yet considered is the rotation of the locus about the origin O. This will be taken up in the next chapter in the discussion of a new system of coördinates known as *polar coördinates.* The rotation of any locus about the X-axis or about the Y-axis is readily accomplished, however, as previously explained. For substituting $(-x)$ for x changes every point that is to the right of the Y-axis to a point to the left thereof, and *vice versa.* It is equivalent, therefore, to a rotation of the locus about the Y-axis. Likewise, substituting $(-y)$ for y rotates any locus 180° about the X-axis. It is preferable, however, to speak of the locus formed in this way as the *reflection* of the original curve in the y-axis or in the x-axis, as the case may be.

39. Roots of Functions. The **roots** or **zeros** of a function are the values of the argument for which the corresponding value of the function is zero. Thus, 2 and 3 are roots of the function $x^2 - 5x + 6$, for substituting either number for x causes the function to be zero. The roots of $x^2 - x - 6$ are $+3$ and -2. The roots of $x^3 - 6x^2 + 11x - 6$ are 1, 2, 3.

The word root, used in this sense, has, of course, an entirely different significance from the same word in "square root," "cube root," etc. But the roots of the function $x^2 - 5x - 6$ are also the roots of the *equation* $x^2 - 5x - 6 = 0$.

In the graph of the cubic function $y = x^3 - x$ in Fig. 42, the curve crosses the X-axis at $x = -1$, $x = 0$, and $x = 1$. These are the values of x that make the function $x^3 - x$ zero, and are, of course, the roots of the function $x^3 - x$. No matter what the function may be, it is obvious that the intercepts on the X-axis, as OA, OB, Fig. 42, must represent the roots of the function.

Exercises

1. From the curve $y = x^2$ sketch the curves $y - 4 = x^2$; $y = 4x^2$; $4y = x^2$; $y = (x - 4)^2$.

2. Sketch $y = x^3/2$; $y = x^3 - 1/4$; $y = x^3/2 - 4$; $y = \frac{(x-3)^2}{2}$.

3. Sketch the curves $y = \sqrt{x}$; $y = \sqrt[3]{x}$; $y = 2\sqrt{x}$; $y = \sqrt{x-2}$; $y - 2 = \sqrt{x-2}$, and $y = \sqrt[3]{x-3}$.

4. Sketch the curves $y^2 = (x - 3)^3$; $(y - 2)^2 = x^3$, and $(y - 2)^2 = (x - 3)^3$.

5. Graph $y_1 = x$ and $y_2 = x^3$ and thence $y = x + x^3$.

6. Find the roots of $x^2 - 6x + 8 = 0$, from the graph of $y = x^2 - 6x + 8$.

7. Find the roots of the functions $x^2 - a^2$ and $x^4 - a^4$.

8. Compare the curves $y = x^3$ and $y = -x^3$; $y = x^2$ and $-y = x^2$; $y = 2x + 3$ and $y = -2x + 3$.

9. Graph $y_1 = x$ and $y_2 = 1/x$ and thence $y = x + 1/x$.

10. Compare $y = 1/x$, $y = 1/(x - 2)$, $y = 1/(x + 3)$.

11. Compare $y = 1/x$, $y = 1/(2x)$, $y = 2/x$.

40. *Graphical Construction of Power Functions and of other Functions.[1] The graphical computation of products and quo-

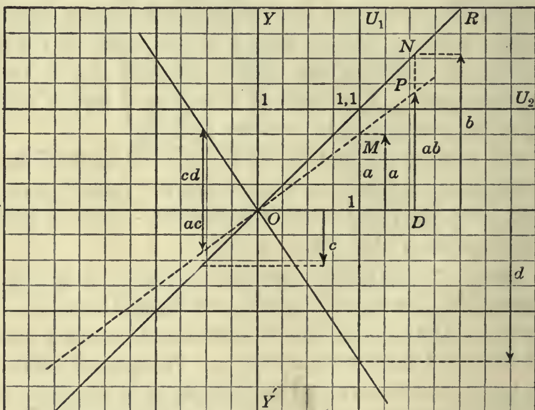


FIG. 45.—Construction of an Ordinate Equal to the Product of Two Given Ordinates.

tients, etc., explained in §7, may be applied to the construction of the power functions. For this purpose it is desirable to elaborate slightly the previous method so as to provide for finding products, etc., of lines that are parallel to each other, instead of at right angles as OA and $1B$, Fig. 9.

[1] The remainder of this chapter (except the review exercises) may be omitted without loss of continuity.

The constructions can be carried out on plain paper by first drawing the axes, the unit lines and the line $y = x$, without the use of scales or measuring device of any sort. The work is more rapidly done, however, on squared paper, as then the use of a T-square and triangle may be dispensed with. A unit of measure equal to 2 inches or 4 cm. will be found convenient for work on standard letter paper $8\frac{1}{2} \times 11$ inches.

Note that the following constructions give both the magnitude and the proper algebraic sense of the results.

(1) *To construct an ordinate equal to the product of two ordinates:* Let XX', YY', Fig. 45, be the axes, U_1, U_2 the unit lines, and OR the line $y = x$, which

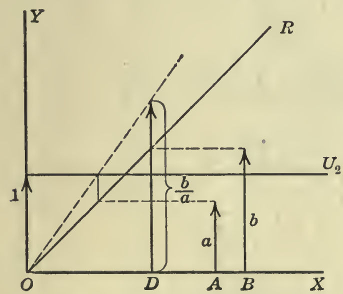


FIG. 46.—Construction of an Ordinate Equal to the Quotient of Two Given Ordinates.

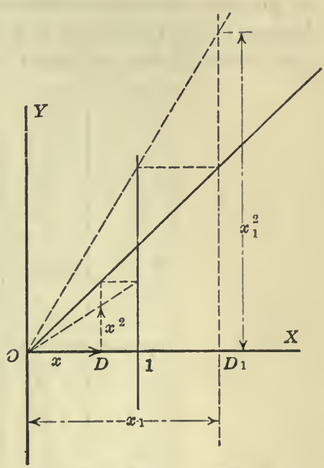


FIG. 47.—Construction of an Ordinate Equal to the Square of a Given Ordinate.

we shall call the **reflector**. Let a and b be two ordinates whose product is required. Move one of the two given ordinates as b until, in the position ND, its end touches the reflector OR. Move the second of the two ordinates to the position $1M$ on the unit line U_1. Draw OMP. The point P at which DN is cut by OM (produced if necessary) determines DP, which is the product $a \times b$. This result follows by similar triangles from the proportion

$$DP : 1M = OD : O1$$

Substituting a for $1M$ and b for OD ($= DN = b$) and unity for $O1$, we obtain

$$DP : a = b : 1$$

or

$$DP = a \times b$$

The same diagram shows the construction of the products $c \times d$ and $a \times c$ for cases in which one or both of the factors are negative.

Note that by the above construction the ordinate representing the product is always located at a particular place, D, at which the abscissa of the product $a \times b$ is either equal to a or to b, depending upon which of the ordinates was moved to the reflector OR.

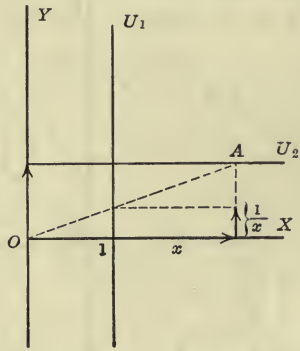


Fig. 48.—Construction of the Reciprocal of x.

(2) *To construct an ordinate equal to the quotient of two ordinates:* This is done by use of the second unit line U_2 as shown in Fig. 46. The ordinate representing the quotient is located at D where OD equals the dividend b.

(3) The special case of (1) when $a = b$ leads to the construction of x^2 as shown in Fig. 47. The figure shows the construction of x^2 at D where $OD = x$ and of $x_1{}^2$ at D_1 where $OD_1 = x_1$.

(4) The special case of (2) where $b = 1$ leads to the construction of $1/x$ as shown in Fig. 48.

(5) *To construct the graph of $y = x^2$,* it is merely necessary to make repeated applications of (3) to the successive ordinates of

the line $y = x$, as shown in Fig. 49. Thus from any point A of $y = x$ move horizontally to the unit line U_1 locating B, then if OB meets DA at P, P is a point of the curve $y = x^2$. The figure shows the construction for a number of points, lettered A_1, A_2, A_3, . . .

(6) *To construct the graph of* $y = x^3$, first cut out a pattern of $y = x^2$ of heavy paper, marking upon it the lines OY and U_2

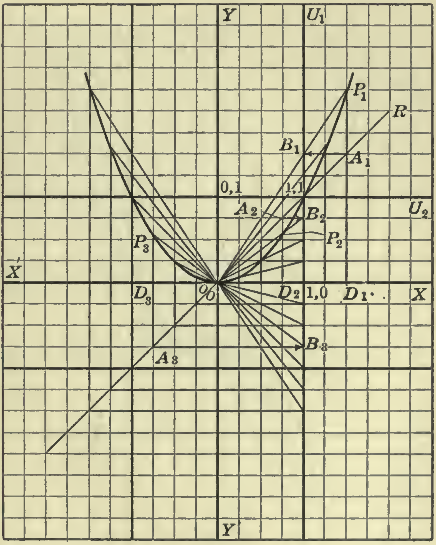


FIG. 49.—Construction of the Curve $y = x^2$ from the curve $y = x$.

By means of this pattern draw the curve $y = x^2$ upon a fresh sheet of paper as shown in Fig. 50. Then multiply each ordinate of $y = x^2$ by x by moving it horizontally from any point A of $y = x^2$ to the unit line U_1 at B, then locating P on DA by drawing OB until it cuts DA at P. The result is the cubical parabola $P_1P_2P_4P'_4P'_2$.

(7) *To draw the hyperbola* $y = 1/x$, make repeated application of (4) above to successive values of x. To draw $y = 1/x^2$, repeat division by x to the ordinates of $y = 1/x$, etc.

(8) *To construct the graph of* $y = x^{3/2}$: First, from the pattern of $y = x^2$ draw the curve $y = \sqrt{x}$. From a pattern draw the curve $y = x^3$ upon the same axes. Then from any point A_1 of $y = x^{1/2}$ proceed horizontally to B_1 on the reflector; then vertically to C_1 on the curve $y = x^3$, then horizontally to P_1 on the ordinate DA_1 first taken. Then P_1 is on the curve $y = x^{3/2}$. For, call $DA_1 = y_1$; $HC_1 = y_2$; $DP_1 = y$; $OD = x$; $OH = x_2$. Then by construction (Fig. 51)

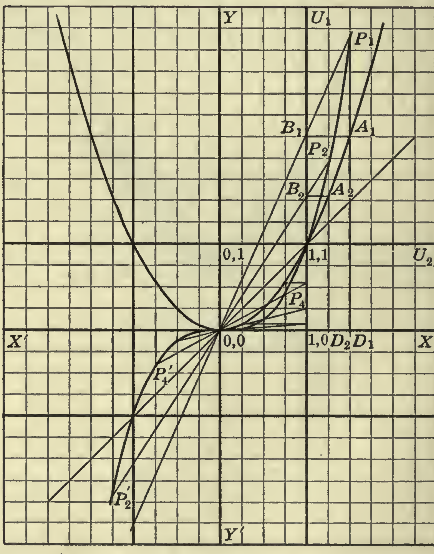


FIG. 50.—Construction of the Curve $y = x^3$ from the curve $y = x^2$.

$$OH = DA = y_1 = x^{1/2} \tag{1}$$

$$DP_1 = y = HC_1 = y_2 = x_2^3 \tag{2}$$

But,

$$y_1 = x_2 \tag{3}$$

Hence, by (3) and (2):

$$y = y_1^3$$

and by (1)

$$y = (x^{1/2})^3 = x^{3/2} \tag{5}$$

(9) *Function of a function:* The construction and reasoning just given applies to a much more general case. Thus if the curve OA_1, Fig. 51, has the equation

$$y = f(x)$$

and if the curve OC_1 has the equation

$$y = F(x)$$

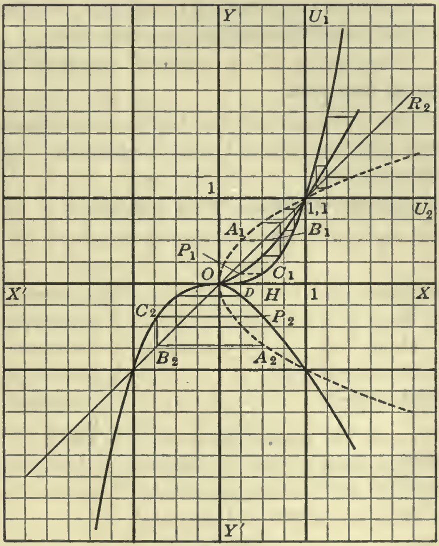


Fig. 51.—Construction of the Semi-cubical Parabola $y = x^{3/2}$ from $y = x^3$ and $y = x^{1/2}$

then the curve OP_1 has the equation

$$y = F[f(x)]$$

Thus, if OA_1 be the curve

$$y = 1 - x^2$$

and OC_1 the curve

$$y = x^{5/2}$$

then OP_1 is the curve

$$y = (1 - x^2)^{5/2}$$

For constructions of the function

$$y = a + a_1x + a_2x^2 + \ . \ \ . \ \ . + a_nx^n$$

see "Graphical Methods" by Carl Runge, Columbia University Press, 1912.

Miscellaneous Exercises

1. Define a function. Explain what is meant by a discontinuous function. Give practical illustrations.

2. Define an algebraic function; rational function; fractional function. Give practical illustrations in each case.

3. Give an illustration of a rational integral function; of a rational fractional function.

4. Write a short discussion of the Cartesian method of locating a point. Explain what is meant by such terms as "axis," "x of a point," "quadrant," etc.

5. What is meant by the *locus* of an equation?

6. Write the equations of the lines determined by the following data:

(a)	slope 2	Y-intercept 5
(b)	slope -2	Y-intercept 5
(c)	slope 2	Y-intercept -5
(d)	slope -2	Y-intercept -5
(e)	slope -2	X-intercept 4

7. Make two suitable graphs upon a single sheet of squared paper from the following data giving the highest and lowest average closing price of twenty-five leading stocks listed on the New York Stock Exchange for the years given in the table:

Year	Highest	Lowest
1913	94.56	79.58
1912	101.40	91.41
1911	101.76	86.29
1910	111.12	86.32
1909	112.76	93.24
1908	99.04	67.87
1907	109.88	65.04
1906	113.82	93.36
1905	109.05	90.87
1904	97.73	70.66
1903	98.16	68.41
1902	101.88	87.30

Should smooth curves be drawn through the points plotted from this table?

8. Define a parabolic curve. What is the equation of the parabola? Of the cubical parabola? Of the semi-cubical parabola?

9. What is the definition of an hyperbolic curve? Of the rectangular hyperbola?

10. Draw on a sheet of coördinate paper the lines $x = 0$, $x = 1$, $x = -1$, $y = 0$, $y = 1$, $y = -1$. Shade the regions in which the hyperbolic curves lie with vertical strokes; and those in which the parabolic curves lie with horizontal strokes. *Write down* all that the resulting figure tells you.

11. Consider the following: $y = x^2$, $y = x^{-3}$, $y = \sqrt[3]{x^4}$, $xy = -1$, $y = -x^3$, $y^3 = x^4$, $y^4 = x^2$, $xy = 1$, $x^3 = -y^2$, $x^4 = -y^2$. Which are increasing functions of x in the first quadrant? For which does the slope of the curve increase in the first quadrant? For which does the slope of the curve decrease in the first quadrant?

12. Which of the curves of exercise 11 pass through (0, 0)? Through (1, 1)? Through (−1, −1)?

13. Find the vertex of the curve $y = x^2 - 24x + 150$.

NOTE: The lowest point of the parabola $y = x^2$ may be called the *vertex*.

SUGGESTION: It is necessary to put the equation in the form $y - b = (x - a)^2$. This can be done as follows: Add and subtract 144 on the right side of the equation, obtaining

$$y = x^2 - 24x + 144 - 144 + 150$$

or,

$$y = (x - 12)^2 + 6$$

or,

$$y - 6 = (x - 12)^2$$

Then this is the curve $y = x^2$ translated 12 units to the right and 6 units up. Since the vertex of $y = x^2$ is at the origin, the vertex of the given curve must be at the point (12, 6).

14. Find the vertex of the parabola $y = x^2 - 6x + 11$.

15. Find the vertex of $y = x^2 + 8x + 1$.

16. Find the vertex of $4 + y = x^2 - 7x$.

17. Find the vertex of $y = 9x^2 + 18x + 1$.

18. Translate $y = 4x^2 - 12x + 2$ so that the equation may have the form $y = 4x^2$.

CHAPTER III

THE CIRCLE AND THE CIRCULAR FUNCTIONS

41. Equation of the Circle. In rectangular coördinates the abscissa x, and the ordinate y, of any point P (as OD and DP, Fig. 52) form two sides of a right triangle whose hypotenuse squared is $x^2 + y^2$. If the point P move in such manner that the length of this hypotenuse remains fixed, the point P describes a circle whose center is the origin (see Fig. 52). The equation of this circle is, therefore:

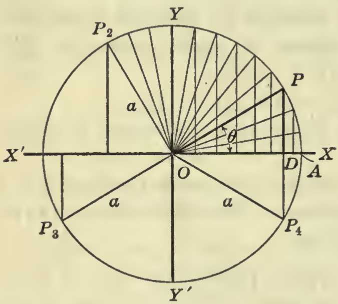


FIG. 52.—The Definition of the Circular Functions.

$$\mathbf{x^2 + y^2 = a^2} \tag{1}$$

if a stand for OP, Fig. 52, namely the fixed length of the hypotenuse, or the radius of the circle.

It is sometimes convenient to write the equation of the circle solved for y in the form

$$\mathbf{y = \pm\sqrt{a^2 - x^2}} \tag{2}$$

This gives, for each value of x, the two corresponding equal and opposite ordinates.

To translate the circle of radius a so that its center shall be the point (h, k), it is merely necessary to write

$$\mathbf{(x - h)^2 + (y - k)^2 = a^2} \tag{3}$$

This is the general equation of any circle in the plane xy, for it locates the center at any desired point and provides for any desired radius a.

Exercises

1. Write the equations of the circles with center at the origin having radii 3, 4, 11, $\sqrt{2}$ respectively.

2. Write the equation of each circle described in exercise 1 in the form $y = \pm\sqrt{a^2 - x^2}$.

3. Which of the following points lie on the circle $x^2 + y^2 = 169$ (5, 12), (0, 13), (− 12, 5), (10, 8), (9, 9), (9, 10)?

4. Which of the following points lie inside and which lie outside of the circle $x^2 + y^2 = 100$: (7, 7), (10, 0), (7, 8), (6, 8), (− 5, 9), (− 7, − 8), (2, 3), (10, 5), ($\sqrt{40}$, $\sqrt{50}$), ($\sqrt{19}$, 9)?

42. The Equation, $\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0}$ (1)
may be put in the form (3). For it may be written

$$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c,$$

or,

$$(x + g)^2 + (y + f)^2 = (\sqrt{g^2 + f^2 - c})^2 \tag{2}$$

which represents a circle of radius $\sqrt{g^2 + f^2 - c}$ whose center is at the point $(-g, -f)$. In case $g^2 + f^2 - c < 0$, the radical becomes imaginary, and the locus is not a real circle; that is, coördinates of no points in the plane xy satisfy the equation. If the radical be zero, the locus is a single point.

43. *Any equation of the second degree, in two variables, lacking the term xy and having like coefficients in the terms x^2 and y^2, represents a circle, real, null or imaginany.* The general equation of the second degree in two variables may be written:

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0 \tag{3}$$

for, when only two variables are present, there can be present three terms of the second degree, two terms of the first degree, and one term of the zeroth degree. When $a = b$ and $h = 0$ this reduces to (1) above on dividing through by a.

Exercises

Find the centers and the radii of the circles given by the following equations:

1. $x^2 + y^2 = 25$. Also determine which of the following points are on this circle: (3, 4), (5, 5), (4, 3), (−3, −4), (−3, 4), (5, 0), (2, $\sqrt{21}$).

2. $x^2 + y^2 = 16$.

3. $x^2 + y^2 - 4 = 0$.

4. $x^2 + y^2 - 36 = 0$.

5. $x^2 + y^2 + 2x = 0$.

6. $y = \pm\sqrt{169 - x^2}$. Also find the slope of the diameter through the point (5, 12). Find the slope of the tangent at (5, 12).

7. $9 - x^2 - y^2 = 0$.

8. $x^2 + y^2 - 6y = 16$.

9. $x^2 - 2x + y^2 - 6y = 15$.

10. $(x + a)^2 + (y - b)^2 = 50$.

11. $x^2 + y^2 + 6x - 2y = 10$.

12. $x^2 + y^2 - 4x + 6y = 12$.

13. $x^2 + y^2 - 4x - 8y + 4 = 0$.

14. $3x^2 + 3y^2 + 6x + 12y - 60 = 0$.

15. Is $x^2 + 2y^2 + 3x - 4y - 12 = 0$ the equation of a circle? Why?

16. Is $2x^2 + 2y^2 - 3x + 4y - 8 = 0$ the equation of a circle? Why?

44. Angular Magnitude. By the **magnitude of an angle** is meant the amount of rotation of a line about a fixed point. If a line OA rotate in the plane XY about the fixed point O to the position OP, the line OA is called the **initial side** and the line OP is called the **terminal side** of the angle AOP. The notion of angular magnitude as introduced in this definition is more general than that used in elementary geometry. There are two new and very important consequences that follow therefrom:

(1) Angular magnitude is unlimited in respect to size—that is, it may be of any amount whatsoever. An angular magnitude of 100 right angles, or twenty-five complete rotations is quite as possible, under the present definition, as an angle of smaller amount.

(2) Angular magnitude exists, under the definition, in two opposite senses—for rotation may be **clockwise** or **anti-clockwise.** As is usual in mathematics, the two opposite senses are distinguished by the terms **positive** and **negative.** In Fig. 53, AOP_1, AOP_2, AOP_3, AOP_4 are positive angles. In designating an angle its initial side is always named first. Thus, in Fig. 53, AOP_1 designates a positive angle of initial side OA. P_1OA designates a negative angle of initial side OP_1.

In Cartesian coördinates, OX is usually taken as the initial line for the generation of angles. If the terminal side of any angle falls within the second quadrant, it is said to be an "angle of the second quadrant," etc.

Two angles which differ by any multiple of 360° are called **congruent angles.** We shall find that in *certain cases* congruent angles may be substituted for each other without modifying results.

The theorem in elementary geometry, that angles at the center of a circle are proportional to the intercepted arcs, holds obviously for the more general notion of angular magnitude here introduced.

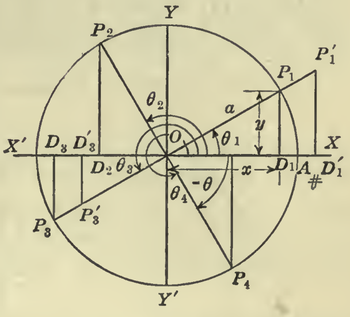


FIG. 53.—Triangles of Reference (OD_1P_1, OD_2P_2, etc.) for Angles θ of Various Magnitude.

45. Units of Measure. Angular magnitude, like all other magnitudes, must be measured by the application of a suitable unit of measure. Four systems are in common use:

(1) **Right Angle System.** Here the unit of measure is the right angle, and all angles are given by the number of right angles and fraction of a right angle therein contained. This unit is familiar to the student from elementary geometry. A practical illustration is the scale of a mariner's compass, in which the right angles are divided into halves, quarters and eighths.

(2) **The Degree System.** Here the unit is the angle corresponding to $\frac{1}{360}$ of a complete rotation. This system, with the sexagesimal sub-divisions (division by 60ths) into **minutes** and **seconds,** is familiar to the student. This system dates back to remote antiquity. It was used by, if it did not originate among, the Babylonians.

(3) **The Hour System.** In astronomy, the angular magnitude about a point is divided into 24 hours, and these into minutes and seconds. This system is familiar to the student from its analogous use in measuring time.

(4) **The Radian or Circular System.** Here the unit of measure

is an angle such that the length of the arc of a circle described about the vertex as center is equal to the length of the radius of the circle. This system of angular measure is fundamental in mechanics, mathematical physics and pure mathematics. It must be thoroughly mastered by the student. The unit of measure in this system is called the **radian.** Its size is shown in Fig. 54.

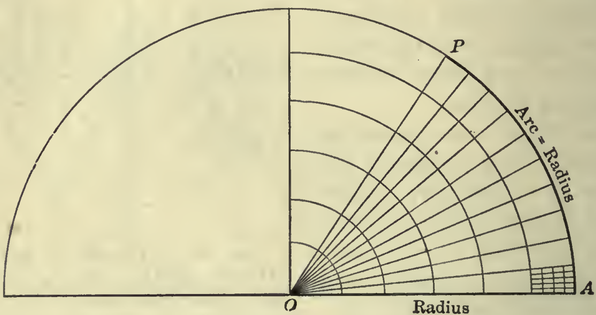


FIG. 54.—Definition of *the Radian.* The Angle *AOP* is *one Radian.*

Inasmuch as the radius is contained 2π times in a circumference, we have the equivalents:

$$2\pi \text{ radians} = 360°.$$

or,

$$1 \text{ radian} = 57° \, 17' \, 44''.8 = 57° \, 17'.7 = 57°.3.$$

$$1 \text{ degree} = 0.01745 \text{ radians}.$$

The following equivalents are of special importance:

$$\text{a straight angle} = \pi \text{ radians}.$$

$$\text{a right angle} = \frac{\pi}{2} \text{ radians}.$$

$$60° = \frac{\pi}{3} \text{ radians}.$$

$$45° = \frac{\pi}{4} \text{ radians}.$$

$$30° = \frac{\pi}{6} \text{ radians}.$$

There is no generally adopted scheme for writing angular magnitude in radian measure. We shall use the superior Roman letter "r" to indicate the measure, as for example, $18° = 0.31416^{\mathrm{r}}$.

Since the circumference of a circle is incommensurable with its diameter, it follows that the number of radians in an angle is always incommensurable with the number of degrees in the angle.

The speed of rotating parts, or angular velocities, are usually given either in revolutions per minute (abbreviated "r.p.m.") or in radians per second.

46. Uniform Circular Motion. Suppose the line OP, Fig. 52, is revolving counter-clockwise k^{r} per second, the angle AOP in radians is then kt, t being the time required for OP to turn from the initial position OA. If we call θ the angle AOP, we have $\theta = kt$ as the equation defining the motion. The following terms are in common use:

1. The **angular velocity** of the uniform circular motion is k (radians per second).

2. The **amplitude** of the uniform circular motion is a.

3. The **period** of the uniform circular motion is the number of seconds required for one revolution.

4. The **frequency** of the uniform circular motion is the number of revolutions per second.

Sometimes the unit of time is taken as one minute. Also the motion is sometimes clockwise or negative.

Exercises

1. Express each of the following in radians: 135°, 330°, 225°, 15°, 150°, 75°, 120°. (Do *not* work out in decimals; use π).

2. Express each of the following in degrees and minutes: 0.2^{r}, $\pi^{\mathrm{r}}/5$, $\frac{3}{2}\pi^{\mathrm{r}}$, $\frac{5}{4}\pi^{\mathrm{r}}$.

3. How many revolutions per minute is 20 radians per second?

4. The angular velocity, in radians per second, of a 36-inch automobile tire is required, when the car is making 20 miles per hour.

5. What is the angular velocity in radians per second of a 6-foot drive-wheel, when the speed of the locomotive is 50 miles per hour?

6. The frequency of a cream separator is 6800 r.p.m. What is its period, and velocity in radians?

7. A wheel is revolving uniformly 30^r per second. What is its period, and frequency?

8. The speed of the turbine wheel of a 5-h.p. DeLaval steam turbine is 30,000 r.p.m. What is the angular velocity in radians per second?

47. The Circular or Trigonometric Functions. To each point on the circle $x^2 + y^2 = a^2$ there corresponds not only an abscissa and an ordinate, but also an angle $\theta < 360°$, as shown in Figs. 52 and 53. This angle is called the **direction angle** or **vectorial angle** of the point P. When θ is given, x, y and a are not determined, but the *ratios* y/a, x/a, y/x, and their reciprocals, a/y, a/x, x/y *are determined.* Hence **these ratios are, by definition, functions of** θ. They are known as the **circular** or **trigonometric functions** of θ, and are named and written as follows:

Function of θ.	Name.	Written.
y/a.	sine of θ.	sin θ.
x/a.	cosine of θ.	cos θ.
y/x.	tangent of θ.	tan θ.
x/y.	cotangent of θ.	cot θ.
a/x.	secant of θ.	sec θ.
a/y.	cosecant of θ.	csc θ.

The circular functions are usually thought of in the above order: that is, in such order that the first and last, the middle two, and those intermediate to these, are reciprocals of each other.

The names of the six ratios must be carefully committed to memory, They should be committed, using the *names* of x, y, and a as follows:

Ratio.	Written.
ordinate/radius.	sin θ.
abscissa/radius.	cos θ.
ordinate/abscissa.	tan θ.
abscissa/ordinate.	cot θ.
radius/abscissa.	sec θ.
radius/ordinate.	csc θ.

The right triangle POD of sides x, y and a, whose ratios give the functions of the angle XOP, is often called the **triangle of reference** for this angle. It is obvious that the *size* of the triangle of reference has no effect of itself upon the value of the functions of the angle. Thus in Fig. 53 either P_1OD_1 or $P_1'OD_1'$ may be taken as the triangle of reference for the angle θ_1. Since the triangles are similar we have.

$$\frac{P_1D_1}{OD_1} = \frac{P_1'D_1'}{OD_1'} \qquad \frac{P_1D_1}{OP_1} = \frac{P_1'D_1'}{OP_1'}$$

etc., which shows that identical ratios or trigonometric functions of θ are derived from the two triangles of reference.

48. Elaborate means of computing the six functions have been devised and the values of the functions have been placed in convenient tables for use. The functions are usually printed to 3, 4, 5 or 6 decimal places, but tables of 8, 10 and even 14 places exist. The functions of only a few angles can be computed by elementary means; these angles, however, are especially important.

(1) **The Functions of 30°.** In Fig. 55*a*, if angle AOB be 30°, angle ABO must be 60°. Therefore, constructing the equilateral triangle BOB', each angle of triangle BOB' is 60°, and

$$y = AB = \tfrac{1}{2}\cdot BB' = \tfrac{1}{2}\cdot a$$

Therefore,

$$\sin 30^\circ = \frac{y}{a} = \frac{\frac{1}{2}\cdot a}{a} = 1/2$$

Also:

$$OA = \sqrt{\overline{OB}^2 - \overline{AB}^2} = \sqrt{a^2 - \tfrac{1}{4}a^2} = \tfrac{1}{2}a\sqrt{3}$$

Therefore,

$$\sin 30^\circ = 1/2.$$

$$\cos 30^\circ = \frac{\frac{1}{2}a\sqrt{3}}{a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{\frac{1}{2}a}{\frac{1}{2}a\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cot 30^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3}$$

$$\sec 30^\circ = \frac{1}{\cos 30^\circ} = \frac{2\sqrt{3}}{3}$$

$$\csc 30^\circ = \frac{1}{\sin 30^\circ} = 2$$

(2) **Functions of 45°.** In the diagram, Fig. 55*b*, the triangle OAB is isosceles, so that $y = x$, and $a^2 = x^2 + y^2 = 2x^2$. It follows that $a = x\cdot\sqrt{2} = y\cdot\sqrt{2}$.

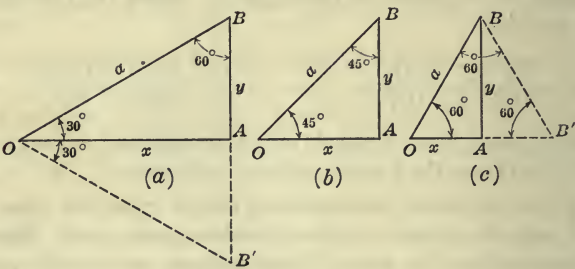


Fig. 55.—Triangles of Reference for Angles of 30°, 45° and 60°.

Therefore:

$$\sin 45° = \frac{y}{y\cdot\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{x}{x\cdot\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{y}{x} = 1.$$

$$\cot 45° = \frac{1}{\tan 45°} = 1$$

$$\sec 45° = \frac{1}{\cos 45°} = \sqrt{2}$$

$$\csc 45° = \frac{1}{\sin 45°} = \sqrt{2}$$

(3) **Functions of 60°.** In the diagram, Fig. 55*c*, construct the equiangular triangle OBB'; then it is seen that, as in case (1) above,

$$OA = \tfrac{1}{2}\cdot OB' = \tfrac{1}{2}\cdot a$$

and

$$y = \sqrt{a^2 - \tfrac{1}{4}\cdot a^2} = \tfrac{1}{2}\cdot a\sqrt{3}$$

Therefore:

$$\sin 60° = \frac{\tfrac{1}{2}\cdot a\cdot\sqrt{3}}{a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{\frac{1}{2}\cdot a}{a} = 1/2$$

$$\tan 60^\circ = \frac{\frac{1}{2}\cdot a\cdot\sqrt{3}}{\frac{1}{2}a} = \sqrt{3}$$

$$\cot 60^\circ = \frac{1}{\tan 60^\circ} = \frac{\sqrt{3}}{3}$$

$$\sec 60^\circ = \frac{1}{\cos 60^\circ} = 2$$

$$\csc 60^\circ = \frac{1}{\sin 60^\circ} = \frac{2\sqrt{3}}{3}$$

49. Graphical Computation of Circular Functions. Approximate determination of the numerical values of the circular functions of any given angle may be made graphically on ordinary coördinate paper. Locate the vertex of the angle at the intersection of any two lines of the squared paper, form $M1$. Let the initial side of the angle coincide with one of the rulings of the squared paper and lay off the terminal side of the angle by means of a protractor. If the sine or cosine is desired, describe a circle about the vertex of the angle as center using a radius appropriate to the scale of the squared paper—for example, a radius of 5 cm. on coördinate paper ruled in centimeters and fifths (form $M1$) permits direct reading to 1/25 of the radius a and, by interpolation, to 1/100 of the radius a. The abscissa and ordinate of the point of intersection of the terminal side of the angle and the circle may then be read and the numerical value of sine and cosine computed by dividing by the length of the radius.

If the numerical value of the tangent or cotangent be required, the construction of a circle is not necessary. The angle should be laid off as above described, and a triangle of reference constructed. To avoid long division, the abscissa of the triangle of reference may be taken equal to 50 or 100 mm. for the determination of the tangent and the ordinate may be taken equal to 50 or 100 mm. for the determination of the cotangent.

The following table (Table III) contains the trigonometric functions of acute angles for each 10° of the argument.

TABLE III

Natural Trigonometric Functions to Two Decimal Places

θ°	θ^r	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0	0.00	0.00	1.00	0.00	∞	1.00	∞
10	0.17	0.17	0.98	0.18	5.67	1.02	5.76
20	0.35	0.34	0.94	0.36	2.75	1.06	2.92
30	0.52	0.50	0.87	0.58	1.73	1.15	2.00
40	0.70	0.64	0.77	0.84	1.19	1.31	1.56
50	0.87	0.77	0.64	1.19	0.84	1.56	1.31
60	1.05	0.87	0.50	1.73	0.58	2.00	1.15
70	1.22	0.94	0.34	2.75	0.36	2.92	1.06
80	1.40	0.98	0.17	5.67	0.18	5.76	1.02
90	1.57	1.00	0.00	∞	0.00	∞	1.00

The most important of these results are placed in the following table:

	0°	30°	45°	60°	90°
Sine	0	1/2	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	1/2	0
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞
$\sqrt{2} = 1.4142$			$\sqrt{3} = 1.7321$		

Exercises

1. Find by graphical construction all the functions of 15°.

NOTE.—A protractor is not needed as angles of 45° and 30° may be constructed.

2. Find tan 60°. Compare with the value found above in §48.

3. Lay off angles of 10°, 20°, 30°, 40°, with a protractor and determine graphically the sine of each angle, and record the results in a suitable table.

4. Find the sine, cosine, and tangent of 75°.

5. Which is greater, sec 40° or csc 50°?

6. Determine the angle whose tangent is 1/2.

7. Find the angle whose sine is 0.6.

8. Which is greater, sin 40° or 2·sin 20°?

9. Does an angle exist whose tangent is 1,000,000? What is its approximate value?

50. Signs of the Functions. The circular functions have, of course, the algebraic signs of the ratios that define them. Of the three numbers entering these ratios, the distance or radius a may always be taken as positive. It enters the ratios, therefore as an always signless, or positive number. The abscissa and the ordinate, x and y, have the algebraic signs appropriate to the quadrants in which P falls. The student should determine the signs of the functions in each quadrant, as follows: (See Fig. 53.)

	First quadrant	Second quadrant	Third quadrant	Fourth quadrant
Sine	+	+	−	−
Cosine	+	−	−	+
Tangent	+	−	+	−

Of course the reciprocals have the same signs as the original functions.

The signs are readily remembered by the following scheme:

Sine		Cosine		Tangent	
+	+	−	+	−	+
−	−	−	+	+	−
Cosecant		Secant		Cotangent	

The following scheme is of value in remembering the circular functions and their signs in the different quadrants: Place on the same line the variables and functions of the same algebraic signs, thus:

Ordinate . . y . . sin θ . . csc θ
Abscissa . . x . . cos θ . . sec θ
Slope . . . m . . tan θ . . cot θ

The above scheme associates the signs of the functions with the coördinates (x, y) of the point P and the slope of the line OP for each of its four positions in Fig. 53.

51. Triangles of reference, geometrically similar to those in Fig. 55 for angles of 30°, 45°, and 60° exist in each of the four quadrants, namely, when the hypotenuse and a leg of the triangle of reference in these quadrants are both either parallel or perpendicular to a hypotenuse and leg of the triangle in the first quadrant—then an acute angle of one must equal an acute angle of the other and the triangles must be similar. The numerical values of the functions in the two quadrants are therefore the same. The algebraic signs are determined by properly taking account of the signs of the abscissa and the ordinate in that quadrant. Thus the triangle of reference for 120° is geometrically similar to that for 60°. Hence, $\sin 120° = \frac{\sqrt{3}}{2}$, but $\cos 120° = -1/2$ and $\tan 120° = -\sqrt{3}$.

Exercises

1. The student is to fill in the blanks in the following table with the correct numerical value and the correct sign of each function:

Function	120°	135°	150°	210°	225°	240°	300°	315°	330°
Sin									
Cos									
Tan									
Cot									
Sec									
Csc									

2. Write down the functions of 390° and 405°.

3. The tangent of an angle is 1. What angle $< 360°$ may it be?

4. $\operatorname{Cos} \theta = -1/2$. What two angles $< 360°$ satisfy the equation?

5. $\operatorname{Sec} \theta = 2$. Solve for all angles $< 360°$.

6. $\operatorname{Csc} \theta = -\sqrt{2}$. Solve for $\theta < 360°$.

52. Functions of 0° and 90°. In Fig. 52 let the angle AOP decrease toward zero, the point P remaining on the circumference of radius a. Then y or PD decreases toward zero. Therefore, $\sin 0° = 0$. Also, x or OD increases to the value a, so that the ratio x/a becomes unity, or $\cos 0° = 1$. Likewise the ratio y/x becomes zero, or $\tan 0° = 0$.

The reciprocals of these functions change as follows: As the angle AOP becomes zero, the ratio a/y **increases in value without limit,** *or* **the cosecant becomes infinite.** In symbols (see §23) $\csc 0° = \infty$. Likewise, $\cot 0° = \infty$, but $\sec 0° = 1$.

In a similar way the functions of 90° may be investigated. The results are given in the following table:

Angle	From 0° to 90°	From 90° to 180°	From 180° to 270°	From 270° to 360°
Sin	0 to +1	+ 1 to 0		
Cos	+1 to 0	0 to − 1		
Tan	0 to + ∞	− ∞ to + 1		
Cot	+ ∞ to 0	0 to − ∞		
Sec	+ 1 to + ∞	− ∞ to − 1		
Csc	+ ∞ to +1	+ 1 to + ∞		

The student is to supply the results for the last two columns.

53. Fundamental Relations. The trigonometric functions are not independent of each other. Because of the relation $x^2 + y^2 = a^2$ it is possible to compute the numerical or absolute value of five of the functions when the value of one of them is given. This may be accomplished by means of the fundamental formulas derived below:

Divide the members of the equation:

$$x^2 + y^2 = a^2 \tag{1}$$

by a^2. Then

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$$

or,

$$\sin^2 \theta + \cos^2 \theta = 1 \tag{2}$$

Likewise divide (1) through by x^2: then

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{a}{x}\right)^2$$

or,

$$\sec^2 \theta = 1 + \tan^2 \theta \tag{3}$$

Also divide (1) through by y^2: then

$$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{a}{y}\right)^2$$

or,

$$\csc^2 \theta = 1 + \cot^2 \theta \tag{4}$$

Also, since

$$\frac{\frac{y}{a}}{\frac{x}{a}} = \frac{y}{x}$$

we obtain:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \tag{5}$$

and likewise

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \tag{6}$$

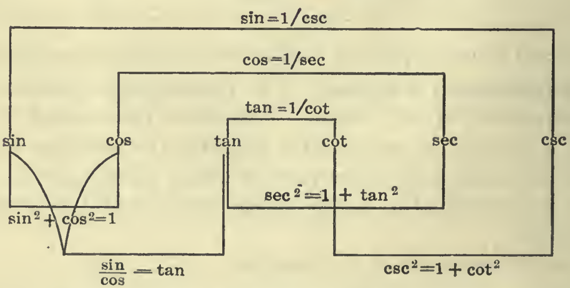


FIG. 56.—Diagram of the Relations between the Six Circular Functions.

Formulas (2) to (6) are the fundamental relations between the six trigonometric functions. The formulas must be committed to memory by the student.

The above relations between the expressions may be illustrated by a diagram as in Fig. 56. The simpler or reciprocal relations are shown by the connecting lines drawn above the functions.

The reciprocal equations and the formulas (2), (3) and (4) are sufficient to express the absolute or numerical value of any function of any angle in terms of any other function of that angle. The algebraic sign to be given the result must be properly selected in each case according to the quadrant in which the angle lies.

Exercises

All angles in the following exercises are supposed to be less than ninety degrees.

1. Sin $\theta = 1/5$. Find cos θ and tan θ.

Draw a right triangle whose hypotenuse is 5 and whose altitude is 1 so that the base coincides with OX. In other words, make $a = 5$ and $y = 1$ in Fig. 57. Calculate $x = \sqrt{25 - 1} = 2\sqrt{6}$ and write down all of the functions from their definitions.

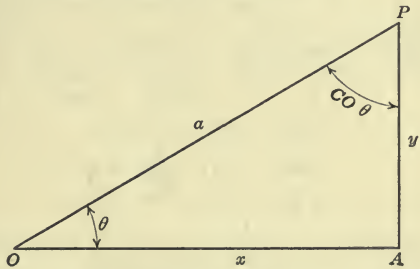


FIG. 57.—Triangle of Reference for θ and Complement of θ.

2. Cos $\theta = 1/3$. Find csc θ.

Take $a = 3$ and $x = 1$ in Fig. 57. Find y and then write down the functions from their definitions.

3. Tan $\theta = 2$. Find sin θ.

Take $x = 1$ and $y = 2$ in Fig. 57, and calculate a and then write down the functions from their definitions.

4. Sec $\theta = 10$. Find csc θ.

Take $a = 10$ and $x = 1$ and compute y.

5. Find the values of all functions of θ if cot $\theta = 1.5$.

6. Find the functions of θ if cos $\theta = 0.1$.

7. Find the values of each of the remaining circular functions in each of the following cases:

(*a*) $\sin\theta = 5/13$. (*d*) $\tan\theta = 3/4$. (*g*) $\tan\theta = m$.

(*b*) $\cos\theta = 4/5$. (*e*) $\sec\theta = 2$. (*h*) $\sin\theta = \dfrac{a}{\sqrt{a^2+b^2}}$

(*c*) $\sec\theta = 1.25$. (*f*) $\tan\theta = 1/3$.

Show that the following equalities are correct:

8. Tan $\theta \cdot \cos\theta = \sin\theta$.

9. Sin $\theta \cdot \cot\theta \cdot \sec\theta = 1$.

10. (Sin $\theta + \cos\theta)^2 = 2 \cdot \sin\theta \cdot \cos\theta + 1$.

11. Tan $\theta + \cot\theta = \sec\theta \cdot \csc\theta$.

12. Express each trigonometric function in terms of each of the others; *i.e.*, fill in all blank spaces in the following table:

	sin	cos	tan	cot	sec	csc
sin	sin					$\frac{1}{\csc}$
cos		cos			$\frac{1}{\sec}$	
tan			tan	$\frac{1}{\cot}$		
cot			$\frac{1}{\tan}$	cot		
sec		$\frac{1}{\cos}$			sec	
csc	$\frac{1}{\sin}$					csc

The following exercises refer to angles $<360°$ of any quadrant:

13. If $\sin\theta = -3/4$ and $\tan\theta$ is positive, find the remaining five functions.

14. If $\cos\theta = 12/13$ and $\sin\theta$ is negative, find the remaining five functions.

15. If $\tan\theta = -\sqrt{3}$ and $\cos\theta$ is negative, find the remaining functions of θ.

16. If $\cos\theta = -1/3$ and $\sin\theta$ is positive, find the remaining functions.

17. If $\tan \theta = 5/12$ and $\sec \theta$ is negative, find the remaining functions of θ.

18. If $\sin \theta = 3/5$ and $\tan \theta$ is negative, find the remaining functions of θ.

54. Functions of Complementary Angles. Complementary angles are defined as two angles whose sum is 90°. **Supplementary angles** are two angles whose sum is 180°.

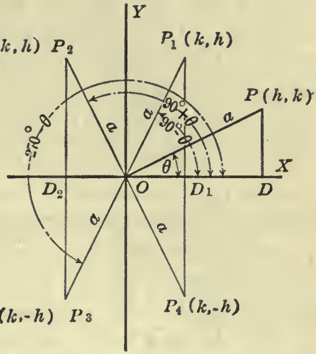


FIG. 58.—Triangles of Reference for θ, and θ combined with an Odd Number of Right Angles.

Let θ be an angle of the first quadrant, and draw the angle $(90° - \theta)$ of terminal side OP_1, as shown in Fig. 58. Let P and P_1 lie on a circle of radius a. Let the coördinates of the point P be (h, k), then P_1 is the point (k, h). Hence $P_1D_1/OP_1 = h/a = \sin(90° - \theta)$. But from the triangle PDO, $h/a = \cos \theta$. Hence

$$\sin(90° - \theta) = \cos \theta$$

Likewise,

$$\tan(90° - \theta) = \cot \theta$$
$$\sec(90° - \theta) = \csc \theta$$

These relations explain the meaning of the words *cosine, cotangent, cosecant,* which are merely abbreviations for *complement's sine, complement's tangent,* etc. Collectively, cosine, cotangent, cosecant are called the **co-functions.** Likewise from Fig. 58:

$$\cos(90° - \theta) = \sin \theta$$
$$\cot(90° - \theta) = \tan \theta$$
$$\csc(90° - \theta) = \sec \theta$$

Later it will be shown that the above relations hold for all values of θ, positive, or negative.

55. Graph of the Sine and Cosine. In rectangular coördinates we can think of the ordinate y of a point as depending for its value upon the abscissa or x of that point by means of the equation $y = \sin x$, provided we think of each value of the abscissa laid off on the X-axis as standing for some amount of angular magnitude. Therefore the equation $y = \sin x$ must possess a graph in rectangular coördinates. In order to produce the graph of $y = \sin x$, it is best to lay off the angular measure x on the X-axis in such a manner that it may conveniently be thought of in either radian or degree measure. If we suppose that a scale of inches and tenths is in the hands of the reader and that a graph is required upon an ordinary sheet of unruled paper of letter size ($8\frac{1}{2} \times 11$ inches), then it will be convenient to let 1/5 inch of the horizontal scale of the X-axis correspond to 10° or to $\pi/18$ radians of angular measure. To

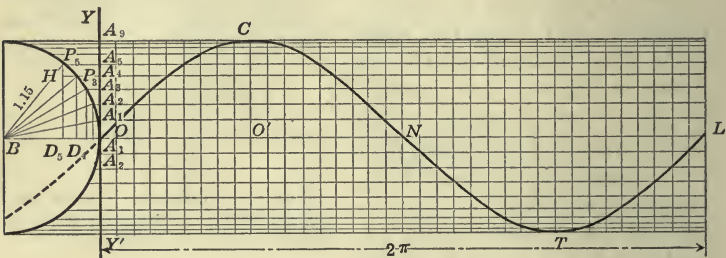


FIG. 59.—Construction of the Sinusoid.

accomplish this, the length of one radian must be 1.15 inches (*i.e.*, $18/5\pi$ inch), which length must be used for the radius of the circle on which the arcs of the angles are laid off. Hence, to graph $y = \sin x$, draw at the left of a sheet of (unruled) drawing paper a circle of radius 1.15 inches, as the circle OPB, Fig. 59. Take O as the origin and prolong the radius BO for the positive portion OX of the X-axis. Subdivide this into 1/5-inch intervals, each corresponding to 10° of angle; eighteen of these correspond to the length π, if the radius BO (1.15 inches) be the unit of measure. Next divide the Y-axis proportionately to $\sin x$ in the following manner: Divide the semicircle into eighteen equal divisions as shown in the figure, thus making the length of each small arc exactly 1/5 inch. The perpendiculars, or ordinates, dropped upon OX from each point of division, divided by the radius a, are the sines of the respective angles. Draw lines parallel to OX through each point of division of this circle. These cut the Y-axis at points A_1, A_2, . . . , such that OA_1, OA_2, . . . are proportional to sin OBP_1, sin OBP_2, sin OBP_3, . . . or in the general case, proportional to sin x (for lack of room only a few of the successive points $P_1, P_2, P_3, \ldots$, of division of the quadrant OP_3P_5, are actually lettered in Fig. 59). These are the successive ordinates corresponding to the abscissas already laid off on OL. The curve is then constructed as follows: First draw vertical lines through the points of division of OX; these, with the horizontal lines already drawn, divide the plane into a large number of rectangles. Starting at O and sketching the diagonals (curved to fit the alignment of the points) of successive "cornering" rectangles, the curve $OCNTL$ is approximated, which is the graph of $y = \sin x$. This curve is called the **sinusoid** or **sine curve.** The curve is of very great importance for it is found to be the *type form* of the fundamental waves of science, such as sound waves, vibrations of wires, rods, plates and bridge members, tidal waves in the ocean, and ripples on a water surface. The ordinary progressive waves of the sea are, however, not of this shape. Using terms borrowed from the language of waves, we may call C the **crest,** N the **node,** and T the **trough** of the sinusoid.

It is obvious that as x increases beyond $2\pi^r$, the curve is repeated, and that the pattern $OCNTL$ is repeated again and again both to the left and the right of the diagram as drawn. Thus it is seen that the sine is a **periodic function** of period $2\pi^r$, or 360°.

The small rectangles lying along the X-axis are *nearly squares.* They would be *exactly* equilateral if the straight line OA_1 was equal to the arc OP_1. This equality is approached as near as we please as the number of corresponding divisions of the circle and of OX is indefinitely increased. In this way we arrive at the notion of the **slope of a curve** in mathematics. In this case we say that the **slope of the sinusoid** at O is $+1$ and at N is -1, and at L is $+1$. We say that the curve cuts the axis at an angle of 45° at O and at an angle of 315° (or, $-45°$ if we prefer) at N. The slope at C and at T is zero.

The curve $y = a \sin x$ is made from $y = \sin x$ by multiplying all of the ordinates of the latter by a. The number a is called the **amplitude** of the sinusoid.

56. Cosine Curve. If O' be taken as the origin, the curve $CNTL$ is the graph of $y = \cos x$. Let the student demonstrate this by showing that the distances BD_1, BD_2, . . . , BD_4 . . . in the semicircle at the left of Fig. 59 go through in reverse order the same sequence of values as P_1D_1, P_2D_2, . . . , and that if the origin be taken at O', the successive ordinates of the sinusoid to the right of $O'C$ are equal to BD_1, BD_2, . . . respectively, and hence are proportional to cos x.

It is best to carry out the construction of the sinusoid upon unruled drawing paper as described above. The curve can readily be drawn, however, upon form $M2$, which is already ruled in 1/5-inch intervals, or upon form $M1$ if the radius of the circle be taken as 2.3 cm. and if 2/5 cm. be used on OX to represent an angle of 10°. A much neater result is obtained when unruled paper is used for the drawing.

57. Complementary Angles. The graph $y_2 = \sin\ (-x)$ is made from $y_1 = \sin x$ by substituting $(-x)$ for x in the function

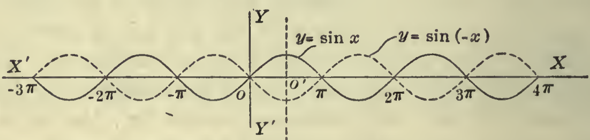


FIG. 60.—Shows the Relation Between $y = \sin x$ and $y = \sin\ (-x)$ and Between $y = \sin\ (90° - x)$ and $y = \cos x$, etc.

sin x; that is, by changing the signs or reversing the direction of all of the abscissas of the sinusoid $y = \sin x$; or, in other words, $y_2 = \sin(-x)$ is the reflection of $y_1 = \sin x$ in the Y-axis. This is merely a special case of the general Theorem I on Loci, **§24.** The former curve has a crest where the latter has a trough and *vice versa,* as is shown by the dotted and full curves in Fig. 60. Now, if the curve $y_2 = \sin\ (-x)$ (the dotted curve in Fig. 60) be translated to the right the distance $\pi/2$, the resulting locus is the cosine curve $y = \cos x$. To translate $y_2 = \sin\ (-x)$ to the right the distance $\pi/2$, the constant $\pi/2$ must be subtracted from the variable x in the equation of the curve, as already learned in the last chapter. Performing this operation we have, for the translated curve,

$$y_2 = \sin\left(-\left[x - \frac{\pi}{2}\right]\right)$$

(Note that $\pi/2$ is subtracted from x and not from $-x$.) Or, removing the brackets,

$$y_2 = \sin\left(\frac{\pi}{2} - x\right)$$

But, as stated above, the curve in its new position is the same as the cosine curve

$$y = \cos x$$

Hence, *for all values of x:*

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad (1)$$

In the same manner it can be proved that $\cos\left(\frac{\pi}{2} - x\right) = \sin x$, and the other results of §**54** follow for all values of x.

58. Trigonometric Functions of Negative Arguments. First compare the curves $y_1 = \sin x$ and $y_2 = \sin(-x)$ as has been done in the preceding section, and as is illustrated by Fig. 60. The curve $y_2 = \sin(-x)$ was described as the reflection of the sinusoid $y_1 = \sin x$ in the Y-axis. It is obvious from the figure, however, that the dotted curve may also be regarded as the reflection of the original curve in the X-axis; for the one has a crest where the other has a trough and the ordinates of the two curves are everywhere of exactly equal length but opposite in direction. This means that $y_2 = -y_1$, or,

$$\sin(-x) = -\sin x \qquad (1)$$

for all values of x.

If the origin be taken at the point O', Fig. 60, the full curve is the graph of $y = \cos x$. In this case the crest of the curve lies above the origin and the curve is symmetrical with respect to the Y-axis. This means that changing x to $(-x)$ in the equation $y = \cos x$ does not modify the locus. Hence we conclude that

$$\cos(-x) = \cos x \qquad (2)$$

for all values of x. Hence by division

$$\tan(-x) = -\tan x \qquad (3)$$

59. Odd and Even Functions. A function that changes sign but retains the same numerical value when the sign of the variable is changed is called an **odd function.** Thus $\sin x$ is an odd function of x, since $\sin(-x) = -\sin x$. Likewise x^3 is an odd function of x, as are all odd powers of x. Geometrically, the graph of an odd function of x is symmetrical with respect to the origin O; that is, if P is a point on the curve, then if the line OP be produced backward through O a distance equal to OP to a point P', then P' lies also on the curve. The branches of $y = x^3$ in the first and third quadrants are good illustrations of this property.

A function of x that remains unaltered (both in sign and numerical value) when the variable is changed in sign, is called an **even function** of x. Examples are $\cos x$, x^2, $x^2 - 3x^4$, . . .

Most functions are neither odd nor even, but mixed, like $x^2 + \sin x$, $x^2 + x^3$, $x + \cos x$, . . .

Exercises

1. Show from (1) and (2) §58 and the relations $\csc x = \dfrac{1}{\sin x}$, $\dfrac{\sin x}{\cos x} = \tan x$, etc., that

$$\begin{aligned}
&(a)\ \csc(-x) = -\csc x\\
&(b)\ \sec(-x) = \sec x\\
&(c)\ \tan(-x) = -\tan x\\
&(d)\ \cot(-x) = -\cot x.
\end{aligned}$$

2. Is $\sin^2 x$ an odd or an even function of x? Is $\tan^3 x$ an odd or an even function of x?

3. Is the function $\sin x + 2 \tan x$ an odd or an even function? Is $\sin x + \cos x$ an odd or an even function of x?

60. The Defining Equations cleared of Fractions. The student should commit to memory the equations defining the trigonometric functions *when cleared of fractions*. In this form the equations are quite as useful as the original ratios. They are written:

$$\begin{aligned}
\mathbf{y} &= \mathbf{a} \sin \theta & \mathbf{x} &= \mathbf{y} \cot \theta\\
\mathbf{x} &= \mathbf{a} \cos \theta & \mathbf{a} &= \mathbf{x} \sec \theta\\
\mathbf{y} &= \mathbf{x} \tan \theta & \mathbf{a} &= \mathbf{y} \csc \theta
\end{aligned}$$

As applied to the right angled triangle, they may be stated in words as follows:

Either leg of a right triangle is equal to the hypotenuse multiplied by the sine of the opposite, or by the cosine of the adjacent, angle.

Either leg of a right triangle is equal to the other leg multiplied by the tangent of the opposite, or by the cotangent of the adjacent, angle.

The hypotenuse of a right triangle is equal to either leg multiplied by the secant of the angle adjacent, or by the cosecant of the angle opposite that leg.

These statements should be committed to memory.

61. Projections. In Fig. 52 the projection of OP in any of its positions, such as $OP_1, OP_2, OP_3, \ldots$, is $OD_1, OD_2, OD_3, \ldots$, or is the abscissa of the point P. Thus for all positions:

$$x = a \cos \theta$$

The sign of x gives the sign, or sense, of the projection. In each case θ is said to be the angle of projection.

The above definition of projection is more general in one respect than that discussed in §**28**. By the present definition the projection of a line is negative if $90° < \theta < 270°$ (read, "if θ is greater than 90° but is less than 270°"). This concept is important and essential in expressing a component of a displacement, of a velocity, of an acceleration, or of a force.

The cosine of θ might have been defined as that proper fraction by which it is necessary to multiply the length of a line in order to produce the projection of the line on a line making an angle θ with it.

Exercises

1. A stretched guy rope makes an angle of 60° with the horizontal. What is the projection of the rope on a horizontal plane? What is the projection of the rope on a vertical plane?

2. Find the lengths of the projections of the line through the origin and the point $(1, \sqrt{3})$ upon the OX and OY axes, if the line is 12 inches long.

3. A force equals 200 dynes. What is its component (projection) on a line making an angle of 135° with the force? On a line making an angle of 120° with the force?

4. A velocity of 20 feet per second is represented as the diagonal of a rectangle the longer side of which makes an angle of 30° with the diagonal. Find the components of the velocity along each side of the rectangle.

5. Show that the projections of a fixed line OA upon all other lines drawn through the point O are chords of a circle of diameter OA. See Fig. 63.

6. Find the projection of the side of a regular hexagon upon the three diagonals passing through one end of the given side, if the numerical value of $\cos 30° = 0.87$, and if each side of the hexagon is 20 feet.

62. Polar Coördinates. In Fig. 61, the position of the point P may be assigned either by giving the x and y of the rectangular coördinate system, or by giving the vectorial angle θ and the distance OP measured along the terminal side of θ. Unlike the distance a used in the preceding work, it is found convenient to give the line OP a **sense** or **direction** as well as length; such a line is called a **vector**. In the present case, it is known as the **radius vector** of the point P, and it is usually symbolized by the letter ρ. The vectorial or direction angle θ and the radius vector ρ are together called the **polar coördinates** of the point P, and the method, as a whole, is known as the **system of polar coördinates.** In Fig. 61 the point P' is located by turning from the fundamental direction OX, called the **polar axis,** through an angle θ and then stepping backward the distance ρ to the point P'; this is, then, the point $(-\rho, \theta)$. P' has also the coördinates (ρ, θ_2), in which $\theta_2 = \theta + 180°$; likewise P_1 is $(+\rho', \theta_1)$ and P'_1 is $(-\rho', \theta_1)$. Thus each point may be located in the polar system of coördinates in two ways, *i.e.*, with either a positive or a negative radius vector. If negative values of θ be used, there are *four* ways of locating a point without using values of $\theta >$ 360°. In giving a point in polar coördinates, it is usual to name the radius vector first and then the vectorial angle; thus (5, 40°) means the point of radius vector 5 and vectorial angle 40°.

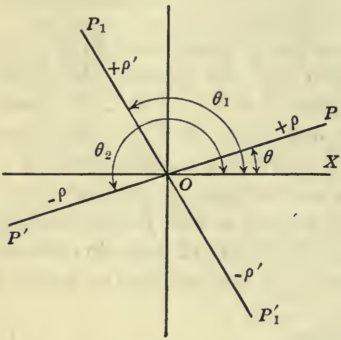


FIG. 61.—Polar Coördinates.

63. Polar Coördinate Paper. Polar coördinate paper (form $M3$) is prepared for the construction of loci in the polar system. A reduced copy of a sheet of such paper is shown in Fig. 62. This plate is graduated in degrees, but a scale of radian measure is given in the margin. The radii proceeding from the pole O meet all of the circles at right angles, just as the two systems of straight lines meet each other at right angles in rectangular coördinate paper. For this reason, both the rectangular and the polar systems are called **orthogonal systems of coördinates.**

We have learned that the fundamental notion of a function implies a table of corresponding values for two variables, one called the argument and the other the function. The notion of a graph

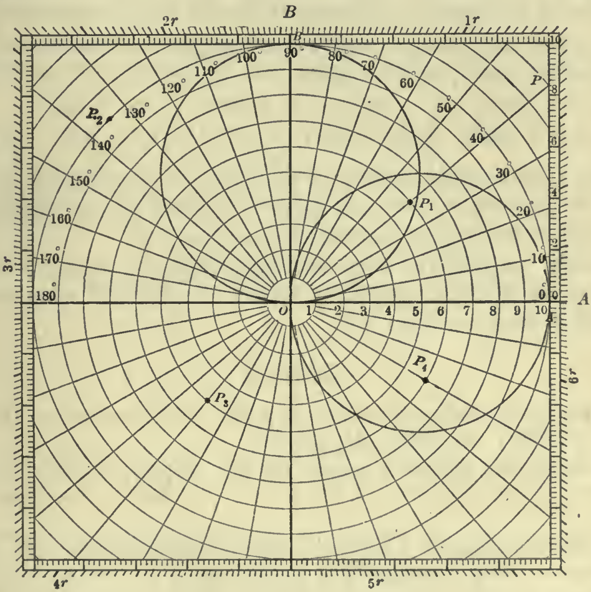


FIG. 62.—Polar Coöradinate Squared Paper. (From $M3$.)

implies any sort of a scheme for a pictorial representation of this table of values. There are three common methods in use: the double scale, the rectangular coördinate paper, and the polar paper. The polar paper is most convenient in case the argument is an angle measured in degrees or in radians. Since in a table of values for a functional relation we need to consider both positive and negative values for both the argument and the function, it is necessary to use on the polar paper the convention already explained. The argument, which is the angle, is measured counter-clockwise if positive and clockwise if negative from the line numbered 0°,

Fig. 62. The function is measured outward from the center along the terminal side of the angle for positive functional values and outward from the center along the terminal side of the angle *produced backward through the center* for negative functional values. In this scheme it appears that four different pairs of values are represented by the same point. This is made clear by the points plotted in the figure. The points P_1, P_2, P_3, P_4 are as follows:

P_1 : (6.0, 40°); (6.0, − 320°); (− 6.0, 220°); (− 6.0, − 140°).
P_2 : (10, 135°); (10, − 225°); (− 10, 315°); (− 10, − 45°).
P_3 : (5, 230°); (5, − 130°); (− 5, 50°); (− 5, − 310°).
P_4 : (6.0, 330°); (6.0, − 30°); (− 6.0, 150°); (− 6.0, − 210°).

The angular scale cannot be changed, but the functional scale can be changed to suit the table of values by multiplying or dividing it by integral powers of ten.

In case the vectorial angle is given in radians, the point may be located on the polar paper by means of a straight edge and the marginal scale on form $M3$.

Exercises

1. Locate the following points on polar coördinate paper; $(1, \pi/2)$; $(2, \pi)$; (3, 60°); (4, 250°); $(2\frac{1}{2}, 1.8\pi)$.

2. Locate the following points: (0, 0°); (1, 10°); (2, 20°); (3, 30°); (4, 40°); (5, 50°); (6, 60°); (7, 70°); . . . (36, 360°). Use 1 cm. = 10 units.

3. The equation of a curve in polar coördinates is $\theta = 2$. Draw the curve. The equation of a second curve is $\rho = 3$. Draw the curve.

Notice that $\rho =$ a constant is a circle with center at O, while $\theta =$ a constant is a straight line through O.

4. Draw the curve $\rho = \theta$ using 2 cm. as unit for ρ. Note that the curve $\rho = \theta$ is a *spiral* while the curve $y = x$ is a straight line.

64. Graphs of $\rho = a \cos \theta$ and $\rho = a \sin \theta$. These are two fundamental graphs in polar coördinates. The equation $\rho = a \cos \theta$ states that ρ is the projection of the fixed length a upon a radial line proceeding from O making a direction angle θ with a, or, in other words, ρ in all of its positions must be the side adjacent to the direction angle θ in a right triangle whose hypotenuse is the finite length a. (See §61.) It must be remembered that the direction angle θ is always measured from the fixed direction OA. Hence, to construct the locus $\rho = a \cos \theta$, draw as many radii vectores as desired, as in Fig. 63. Project on each of these the fixed distance OA or a. This gives OP, or ρ, in numerous positions as shown in the diagram. Since P is by construction the foot of the perpendicular dropped from A upon OP, it is always at the vertex of a right triangle standing on the fixed hypotenuse a, and therefore the point P is on the semicircle AOP; for, from plane geometry a right triangle is always inscribable in a semicircle.

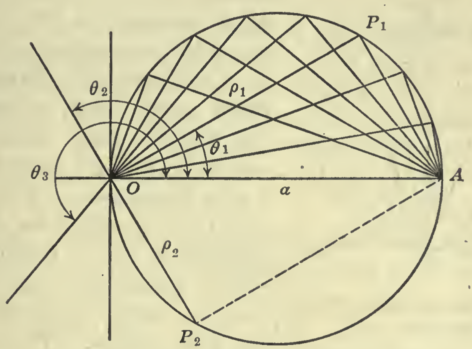


Fig. 63.—The Graph of $\rho = a \cos \theta$.

When θ is in the second quadrant, as θ_2, Fig. 63, the cosine is negative and consequently ρ is also negative. Therefore the point P_2 is located by measuring backward through O. Since, however, ρ_2 is the projection of a through the angle θ_2 (see §**61**), the angle at P_2 must be a right angle. Thus the semicircle OP_2A is described as θ sweeps the second quadrant. When θ is in the third quadrant, as θ_3, the cosine is still negative and ρ is measured backward *to describe the semicircle AP_1O a second time.* As θ sweeps the fourth quadrant, the semicircle OP_2A is described the second time. *Thus the graph in polar coördinates of $\rho = a \cos \theta$ is a circle* **twice drawn** *as θ varies from* 0° to 360°. Once around the circle corresponds to the distance from crest to trough of the "wave" $y = a \cos x$, in Fig. 59 (O' is origin). The second time around the circle corresponds to the distance from trough to crest of the cosine curve. Trough and crest of all the successive "wave lengths" fall at the point A. The nodes are all at O.

The polar representation of the cosine of a variable by means of the circle is more useful and important in science than the Cartesian representation by means of the sinusoid. The ideas here presented must be thoroughly mastered by the student.

The graph of $\rho = a \sin \theta$ is also a circle, but the diameter is the line OB making an angle of 90° with OA, as shown in Fig. 64.

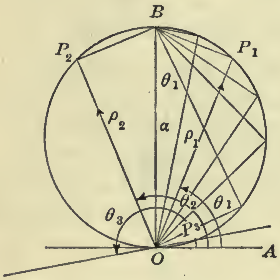


Fig. 64.—The Graph of $\rho = a \sin \theta$.

Since $\rho = a \sin \theta$, the radius vector, as θ increases to 90°, must equal the side lying opposite the angle θ in a right triangle of hypotenuse a. Since angle AOP_1 = angle OBP_1, the point P may be the vertex of any right triangle erected on OB or a as a hypotenuse. The semicircle BP_2O is described as θ increases from 90° to 180°. Beyond 180° the sine is negative, so that the radius vector ρ must be laid off backward for such angles. Thus P_3 is the point corresponding to the angle θ_3, of the third quadrant. As θ sweeps the third and fourth quadrants the circle OP_1BP_2O is described a second time. Therefore, the graph of $\rho = a \sin \theta$ is the circle *twice drawn* of diameter a, and tangent to OX at O. The first time around the circle corresponds to the crest, the second time around corresponds to the trough of the wave or sinusoid drawn in rectangular coördinates. The points corresponding to the nodes of the sinusoid are at O and the points corresponding to the maximum and minimum points are at B.

We have seen that the graph of a function in polar coördinates is a very different curve from its graph in rectangular coördinates. Thus the cosine of a variable if graphed in rectangular coördinates is a sinusoid; but if graphed in polar coördinates the graph is a circle (twice drawn). There is in this case a very great difference in the ease with which these curves can be constructed; the sinusoid requires an elaborate method, while the circle may be drawn at once with compasses. This is one reason why the periodic or sinusoidal relation is preferably represented in the natural sciences by polar coördinates.

65. Graphical Table of Sines and Cosines. The polar graphs of $\rho = a \sin \theta$ and $\rho = a \cos \theta$ furnish the best means of constructing graphical tables of sines and cosines. The two circles passing through O shown on the polar coördinate paper, form $M3$, Fig. 62, are drawn for this purpose. A quantity of this coördinate paper should be in the hands of the student. If the diameter of the sine and cosine circles be called 1, then the radius vector of any point on the lower circle is the cosine of the vectorial angle, and the radius vector of the corresponding point on the upper circle is the sine of the vectorial angle. As there are 50 concentric circles in Form $M3$, it is easy to read the radius vector of a point to 1 /100 of the unit. Thus, from the diagram, we read cos $45° = 0.70$; cos $60° = 0.50$; cos $30° = 0.866$. These results are nearly correct to the third place.

66. Graphical Table of Tangents and Secants. Referring to Fig. 62, it is obvious that the numerical values of the tangents of angles can be read off by use of the uniform scale of centimeters bordering the polar paper (form $M3$). The scale referred to lies just inside of the scale of radian measure, and is numbered 0, 2, 4, . . ., at the right of Fig. 62. Thus to get the numerical value of tan 40° it is merely necessary to call unity the side OA of the triangle of reference OAP, and then read the side $AP = 0.84$; hence tan $40° = 0.84$. To the same scale (*i.e.*, $OA = 1$) the distance $OP = 1.31$, but this is the secant of the angle AOP, whence sec $40° = 1.31$. By use of the circles we find sin $40° = 0.64$ and cos $40° = 0.76$.

In case we are given an angle greater than 45° (but less than 135°) use the horizontal scale through B. Starting from B as zero the distance measured on the horizontal scale is the *cotangent* of the given angle. The tangent is found by taking the reciprocal of the cotangent.

Exercises

Find the unknown sides and angles in the following right triangles. The numerical values of the trigonometric functions are to be taken from the polar paper. The vertices of the triangles are supposed to be lettered A, B, C with C at the vertex of the right angle. The small letters a, b, c represent the sides opposite the angles of the same name.

By **angle of elevation** is meant the angle between a horizontal line and a line to the object, both drawn from the point of observation, when the object lies *above* the horizontal line. The similar angle when the object lies *below* the observer is called the **angle of depression** of the object.

The solution of each of the following problems must be checked. The easiest check is to draw the triangles accurately to scale on form $M1$ and use a protractor.

1. When the altitude of the sun is 40°, the length of the shadow cast by a flag pole on a horizontal plane is 90 feet. Find the height of the pole.

Outline of Solution. Call height of pole a, and length of shadow b. Then $A = 40°$ and $B = 50°$. Hence:

$$a = b \tan 40°$$

Determining the numerical value of the tangent from the polar paper, we find:

$$a = 90 \times 0.84 = 75.6 \text{ ft.}$$

which result, if checked, is the height of the pole. To check, either draw a figure to scale, or compute the hypotenuse c, thus:

$$c = 90 \sec 40°$$

From the polar paper find sec 40°. Then:

$$c = 90 \times 1.31 = 117.9$$

Since $a^2 + b^2 = c^2$, we have $c^2 - b^2 = a^2$, or $(c - b)(c + b) = a^2$. Hence if the result found be correct,

$$(117.9 - 90)(117.9 + 90) = 75.6^2$$
$$5800 = 5715$$

These results show that the work is correct to about three figures, for the sides of the triangle are proportional to the *square roots* of the numbers last given.

2. At a point 200 feet from, and on a level with, the base of a tower the angle of elevation of the top of the tower is observed to be 60°. What is the height of the tower?

3. A ladder 40 feet long stands against a building with the foot of the ladder 15 feet from the base of the wall. How high does the ladder reach on the wall?

4. From the top of a vertical cliff the angle of depression of a point on the shore 150 feet from the base of the cliff is observed to be 30°. Find the height of the cliff.

5. In walking half a mile up a hill, a man rises 300 feet. Find the angle at which the hill slopes.

If the hill does not slope uniformly the result is the *average slope* of the hill.

6. A line 3.5 inches long makes an angle of 35° with OX. Find the lengths of its projections upon both OX and OY.

7. A vertical cliff is 425 feet high. From the top of the cliff the angle of depression of a boat at sea is 16°. How far is the boat from the foot of the cliff?

8. The projection of a line on OX is 7.5 inches, and its projection on OY is 1.25 inches. Find the length of the line, and the angle it makes with OX.

9. A battery is placed on a cliff 510 feet high. The angle of depression of a floating target at sea is 9°. Find the range, or the distance of the target from the battery.

10. From a point A the angle of elevation of the top of a monument is 25°. From the point B, 110 feet farther away from the base of the monument and in the same horizontal straight line, the angle of elevation is 15°. Find the height of the monument.

11. Find the length of a side of a regular pentagon inscribed in a circle whose radius is 12 feet.

12. Proceeding south on a north and south road, the direction of a church tower, as seen from a milestone, is 41° west of south. From the next milestone the tower is seen at an angle of 65° W. of S. Find the shortest distance of the tower from the road.

13. A traveler's rule for determining the distance one can see from a given height above a level surface (such as a plain or the sea) is as follows: "To the height in feet add half the height and take the square root. The result is the distance you can see in miles." Show that this rule is approximately correct, assuming the earth a sphere of radius 3960 miles. Show that the drop in 1 mile is 8 inches, and that the water in the middle of a lake 8 miles in width stands $10\frac{2}{3}$ feet higher than the water at the shores.

14. Observations of the height of a mountain were taken at A and B on the same horizontal line and in the same vertical plane with the top of the mountain. The elevation of the top at A is 52° and at B is 36°. The distance AB is 3500 feet. Find the height of the mountain.

15. The diagonals of a rhombus are 16 and 20 feet, respectively. Find the lengths of the sides and the angles of the rhombus.

16. The equation of a line is $y = \frac{3}{4}x + 10$. Compute the shortest distance of this line from the origin.

17. Find the perimeter and area of $ABCD$, Fig. 65.

67. The Law of the Circular Functions. It will be emphasized in this book that the fundamental laws of exact science are three in number, namely: (1) The power function expressed by $y = ax^n$ where n may be either positive or negative; (2) the harmonic or periodic law $y = a \sin nx$, which is fundamental to all periodically occurring phenomena; and a third law to be discussed in a subsequent chapter. While other important laws and functions arise in the exact sciences, they are secondary to those expressed by the three fundamental relations.

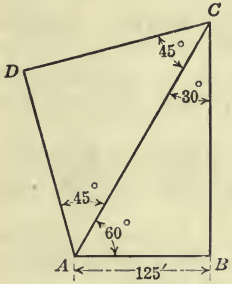


FIG. 65.—Diagram for Exercise 17.

We have stated the law of the power function in the following words (see §**34**): *In any power function, if x change by a fixed multiple, y is changed by a fixed multiple also.* In other words, if x change by a constant factor, y will change by a constant factor also.

Confining our attention to the fundamental functions, sine and cosine, in terms of which the other circular functions can be expressed, we may state their law as follows:[1]

The circular functions, sin θ and cos θ, change periodically in value proportionally to the periodic change in the ordinate and abscissa, respectively, of a point moving uniformly on the circle $x^2 + y^2 = a^2$.

The use of the periodic law in natural science is, of course, very different from that of the power function. The student will find that circular functions similar to $y = a \sin nx$ will be required in order to express properly any phenomena which are recurrent or periodic in character, such as the motion of vibrating bodies, all forms of wave motion, such as sound waves, light waves, electric waves, alternating currents and waves on water surfaces, etc. Almost every part of a machine, no matter how complicated its motions, repeats the original positions of all of the parts at

[1] Chapter X is devoted to a discussion of these fundamental periodic laws.

stated intervals and these recurrent positions are expressible in terms of the circular functions and not otherwise. The student will obtain a most limited and unprofitable idea of the use of the circular functions if he deems that their principal use is in numerical work in solving triangles, etc. The importance of the circular functions lies in the power they possess of expressing natural laws of a periodic character.

68. Rotation of Any Locus. In §36 we have shown that any locus $y = f(x)$ is translated a distance a in the x direction by substituting $(x - a)$ for x in the equation of the locus. Likewise the substitution of $(y - b)$ for y was found to translate the locus the distance b in the y direction. A discussion of the rotation of a locus was not considered at that place, because a displacement of this type is best brought about when the equations are expressed in polar coördinates.

If a table of values be prepared for each of the loci

$$\rho = \cos \theta \tag{1}$$

$$\rho = \cos (\theta_1 - 30°) \tag{2}$$

as follows:

θ	0°	30°	60°	90°	120°	150°	180°	. . .
ρ	1	$\frac{1}{2}\sqrt{3}$	1/2	0	−1/2	$-\frac{1}{2}\sqrt{3}$	−1	. . .

θ_1	30°	60°	90°	120°	150°	180°	. . .
ρ	1	$\frac{1}{2}\sqrt{3}$	1/2	0	−1/2	$-\frac{1}{2}\sqrt{3}$	. . .

and then if the graph of each be drawn, it will be seen that the curves differ only in their location and not at all in shape or size. The reason for this is obvious: The same value of ρ is given by $\theta_1 = 90°$ in the second case as is given by $\theta = 60°$ in the first case, and the same value of ρ is given by $\theta_1 = 60°$ in the second case as is given by $\theta = 30°$ in the first case, etc. The sets of values of ρ in the two cases are identical, but like values correspond to vectorial angles θ differing by 30°. In more general terms the reasoning is that if $(\theta_1 - 30°)$ be substituted for θ in *any* polar equation, then since $(\theta_1 - 30°)$ has been put equal to θ, it follows that $\theta_1 = (\theta + 30°)$, or the new vectorial angle θ_1 is greater than the original θ by the amount 30°. Since all values of θ

in the new locus are increased by 30°, the new locus is the same as the original locus rotated about O (positive rotation) by the amount 30°.

The above reasoning does not depend upon the particular constant angle 30° that happened to be used, but holds just as well if any other constant angle, say α, be used instead. That is, substituting $(\theta_1 - \alpha)$ for θ does not change the size or shape of the locus, but merely rotates it through an angle α in the positive sense. The same reasoning applies also to the general case: If $\rho = f(\theta)$ be the polar equation of any locus, then $\rho = f(\theta_1 - \alpha)$ is the equation of the same locus turned about the fixed point O through the angle α; for if $(\theta_1 - \alpha)$ be everywhere substituted for the vectorial angle θ, θ_1 must be α greater than the old θ. That is, each point is advanced the angular amount α, or turned that amount about the point O. The rotation is positive, or anti-clockwise, if α be positive—thus, substituting $(\theta - 30°)$ for θ in $\rho = a \cos \theta$ turns the circle $\rho = a \cos \theta$ through 30° in the anti-clockwise sense, as is shown in Fig. 66, but substituting $(\theta + 30°)$ for θ in $\rho = a \cos \theta$ turns the circle $\rho = a \cos \theta$ through 30° in the clockwise direction of rotation, as shown in the same figure.

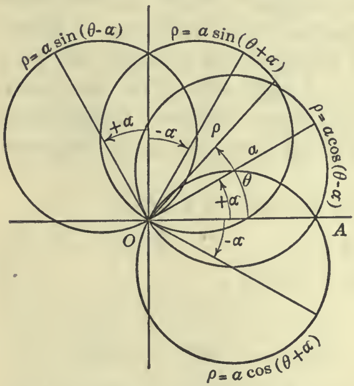


Fig. 66.—Rotation of the Circles $\rho = a \cos \theta$ and $\rho = a \sin \theta$.

The four circles

$$\rho = a \cos (\theta + \alpha) \qquad (3)$$

$$\rho = a \cos (\theta - \alpha) \qquad (4)$$

$$\rho = a \sin (\theta + \alpha) \qquad (5)$$

$$\rho = a \sin (\theta - \alpha) \qquad (6)$$

are shown in Fig. 66. Each has diameter a. The student must carefully distinguish between the constant angle α and the variable angle θ, just as he must distinguish between the constant distance a and the variable vector ρ.

The above result constitutes another of the

Theorems on Loci

IX. *If $(\theta - \alpha)$ be substituted for θ throughout the polar equation of any locus, the curve is rotated through the angle α in the positive sense.*

Note that the substitution is $(\theta - \alpha)$ for θ when the required rotation is through the *positive* angle α, and that the substitution is $(\theta + \alpha)$ for θ when the required rotation is through the *negative* angle α.

The rotation of any locus through any angle is readily accomplished when its equation is given in polar coördinates. Rotations of 180° and 90° are very simple in rectangular coördinates. Let the student select any point P in rectangular coördinates and draw the radius vector OP and the abscissa and ordinate OD and DP; then show that the substitutions $x = -x_1$, $y = -y_1$ will turn OP through 180° about O in the plane xy, and that the substitutions $x = y_1$, $y = -x_1$ will turn OP through 90° about O in the plane xy.

Exercises

Draw the following circles:

1. $\rho = 3 \cos (\theta - 30)°$.

2. $\rho = 3 \cos (\theta + 120°)$.

3. $\rho = 2 \sin (\theta - 45°)$.

4. $\rho = 2 \sin (\theta + 135°)$.

5. $\rho = 4 \cos \left(\theta + \frac{\pi}{3}\right)$.

6. $\rho = 5 \sin \left(\frac{\pi}{2} - \theta)\right)$.

7. Show that $\rho = a \sin \theta$ is the locus $\rho = a \cos \theta$ rotated 90° counter clockwise.

Solution: Write $\rho = a \cos (\theta - 90°)$, then $\rho = a \cos (90° - \theta)$ by (2) §**58**, then $\rho = a \sin \theta$ by §**57**.

69. Polar Equation of the Straight Line. In Fig. 67 let MN be any straight line in the plane and OT be the perpendicular dropped upon MN from the origin O. Let the length of OT be a and let the direction angle of OT be α, where, for a given straight line, a and α are constants. Let ρ be the radius vector of any point

P on the line MN and let its direction angle be θ. Then, by definition,

$$\frac{a}{\rho} = \cos (\theta - \alpha)$$

Therefore the equation of the straight line MN is

$$a = \rho \cos (\theta - \alpha) \tag{1}$$

for it is the equation satisfied by the (ρ, θ) of every point of the line. This is the equation of *any* straight line, for its location is perfectly general. The constants defining the line are the perpendicular distance a upon the given line from O and the direction angle α of this perpendicular. The perpendicular OT or a is called the **normal** to the line MN and the equation (1) is called the **normal equation of the straight line.**

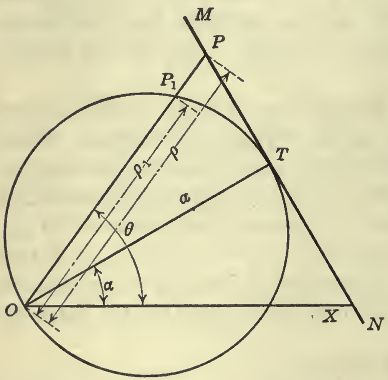


Fig. 67.—The Circle $\rho = a \cos (\theta - \alpha)$ and its Inverse, the line MN or $a = \rho \cos (\theta - \alpha)$.

The equation of the circle shown in the figure is

$$\rho_1 = a \cos (\theta - \alpha) \tag{2}$$

in which ρ_1 represents the radius vector of a point P_1 on the circle. From plane geometry OT or a is a mean proportional between the secant OP and the chord OP_1, or,

$$\rho : a = a : \rho_1$$

or,

$$\rho\rho_1 = a^2 \tag{3}$$

This gives the relation between the radius vector of a point on the line and the corresponding radius vector of a point on the circle. Now if on the radius vector $\rho = OP$, drawn from the fixed origin O to any curve, we lay off a length $OP_1 = \rho_1 = \frac{a^2}{\rho}$ (where a is a constant), then P_1 is said to describe the **inverse** of the given curve with respect to O. In this special case *the circle is the inverse of* *the straight line* and *vice versa.* If $a = 1$ we note that OP_1 and OP are reciprocals of each other.

It is important in mathematics to associate the equation of the circle and the equation of its inverse with respect to O, or the line tangent to it. Thus

$$\rho = 10 \cos\left(\theta - \frac{\pi}{4}\right)$$

is a circle

$$10 = \rho \cos\left(\theta - \frac{\pi}{4}\right)$$

is a straight line tangent to it.

70. Relation between Rectangular and Polar Coördinates. Think of the point P whose rectangular coördinates are (x, y). If the radius vector OP be called ρ and the direction angle be called θ, then the polar coördinates of P are (ρ, θ). Then x and y for any position of P are the projections of ρ through the angle θ, and the angle $(90° - \theta)$, respectively, or,

$$\mathbf{x} = \rho \cos \theta \tag{1}$$

$$\mathbf{y} = \rho \sin \theta \tag{2}$$

These are the equations of transformation that permit us to express the equation of a curve in polar coördinates when its equation in rectangular coördinates is known, and *vice versa.* Thus the straight line $x = 3$ has the equation

$$\rho \cos \theta = 3$$

in polar coördinates. The line $x + y = 3$ has the polar equation

$$\rho \cos \theta + \rho \sin \theta = 3.$$

The circle $x^2 + y^2 = a^2$ has the equation

$$\rho^2 \cdot \cos^2 \theta + \rho^2 \sin^2 \theta = a^2$$

or,

$$\rho^2 = a^2$$

or,

$$\rho = a$$

etc.

To solve equations (1) and (2) for θ, we write

$$\theta = \text{the angle whose cosine is } \frac{x}{\rho}$$

$$\theta = \text{the angle whose sine is } \frac{y}{\rho}$$

The verbal expression "the angle whose cosine is," etc., are abbreviated in mathematics by the notations "$\cos^{-1}$," read "anti-cosine," and "$\sin^{-1}$," read "anti-sine," as follows:

$$\theta = \cos^{-1}(\mathbf{x}/\rho) \tag{3}$$

$$\theta = \sin^{-1}(\mathbf{y}/\rho) \tag{4}$$

Dividing the members of (2) by the members of (1) we obtain $\tan\theta = \frac{y}{x}$, which, solved for θ, we write

$$\theta = \text{the angle whose tangent is } \frac{y}{x}$$

which may be abbreviated

$$\theta = \tan^{-1}(\mathbf{y}/\mathbf{x}) \tag{5}$$

and read "θ = the anti-tangent of y/x."

The value of ρ in terms of x and y is readily written

$$\rho = \sqrt{\mathbf{x}^2 + \mathbf{y}^2} \tag{6}$$

Exercises

1. Write in polar coördinates the equation $x^2 + y^2 + 8x = 0$.
The result is $\rho^2 + 8\rho\cos\theta = 0$, or $\rho = -8\cos\theta$.

2. Write in polar coördinates the equations (*a*) $x^2 + y^2 - 4y = 0$; (*b*) $x^2 + y^2 - 6x - 4y = 0$; (*c*) $x^2 + y^2 - 6y = 4$.

3. Write in polar coördinates the equations (*a*) $x + y = 1$; (*b*) $x + 2y = 1$; (*c*) $x + \sqrt{3}\, y = 2$.

4. Write in rectangular coördinates (*a*) $\rho\cos\theta + \rho\sin\theta = 4$; (*b*) $\rho\cos\theta - 3\rho\sin\theta = 6$.

5. Write in polar coördinates $x^2 + 2y^2 - 4x = 0$.

71. Identities and Conditional Equations. It is useful to make a distinction between equalities like

$$(a - x)(a + x) = a^2 - x^2 \tag{1}$$

which are true for all values of the variable x and equalities like

$$x^2 - 2x = 3 \tag{2}$$

which are true only for certain particular values of the unknown number. When two expressions are equal for all values of the variable for which the expressions are defined, the equality is known as an **identity.** When two expressions are equal only for certain particular values of the unknown number the equality is spoken of as a **conditional equation.** The fundamental formula

$$\sin^2 \phi + \cos^2 \phi = 1$$

is an identity.

$$2 \sin A + 3 \cos A = 3.55$$

is a conditional equation. Sometimes the symbol $\equiv$ is used to distinguish an identity; thus

$$a^3 - x^3 \equiv (a - x)(a^2 + ax + x^2)$$

Exercises

The following exercises contain problems both in the establishment of trigonometric identities and in the finding of the values of the unknown number from trigonometric conditional equations.

The truth of a trigonometric identity is established by reducing each side to the same expression. This usually requires the application of some of the fundamental identities, equations (1) to (5), §53. Facility in the establishment of trigonometric identities is largely a matter of skill in recognizing the fundamental forms and of ingenuity in performing transformations. In verifying the identity of two trigonometic expressions it is best to reduce each expression separately to its simplest form. Unless the student writes the work in two separate columns, transforming the left member alone in one column, and the right member alone in the other column, he is very liable to get erroneous results. *All results should be checked.* The following worked exercises will aid the student.

(a) Prove that

$$(1 - \sin u \cos u)(\sin u + \cos u) \equiv \sin^3 u + \cos^3 u$$

The sum of two cubes is divisible by the sum of the numbers themselves so that after division we have:

$$1 - \sin u \cos u \equiv \sin^2 u - \sin u \cos u + \cos^2 u$$

Since $\sin^2 u + \cos^2 u = 1$, this equation is true and the original identity is established.

(b) Show that

$$\sec^2 x - 1 \equiv \sec^2 x \sin^2 x$$

Substituting $\sec^2 x \equiv \dfrac{1}{\cos^2 x}$ on the right side

$$\sec^2 x - 1 \equiv \frac{\sin^2 x}{\cos^2 x} \equiv \tan^2 x$$

or
$$\sec^2 x \equiv 1 + \tan^2 x$$

which is a fundamental identity.

Solutions to exercises in trigonometric conditional equations similar to exercises 1, 4, 5, 9 below must be **checked.** The necessity for a check is made apparent by the following illustration:

(c) Solve for all angles less than 360°

$$2 \sin x + \cos x = 2 \qquad (1)$$

Transposing and squaring we get:

$$\cos^2 x = 4 - 8 \sin x + 4 \sin^2 x \qquad (2)$$

since $\sin^2 x + \cos^2 x = 1$.

$$1 - \sin^2 x = 4 - 8 \sin x + 4 \sin^2 x \qquad (3)$$

$$5 \sin^2 x - 8 \sin x + 3 = 0 \qquad (4)$$

$$\sin x = 1, \text{ or } 0.6 \qquad (5)$$

$$x = 90°, 37°, \text{ or } 143° \qquad (6)$$

Check: $$2 \sin 90° + \cos 90° = 2 + 0 = 2 \qquad (7)$$

Check: $$2 \sin 37° + \cos 37° = 1.2 + 0.8 = 2 \qquad (8)$$

Does $$2 \sin 143° + \cos 143° = 1.2 - 0.8 = 0.4 = 2? \qquad (9)$$

The last value does not check. The reasons for this will be discussed later in §§**93** and **94.** Therefore the correct solutions are 90° and 37°.

1. Solve for all values of $\theta < 90°$: $6 \cos^2 \theta + 5 \sin \theta = 7$.

Suggestion: Write $6(1 - \sin^2 \theta) + 5 \sin \theta = 7$ and solve the quadratic in $\sin \theta$.

$$6 \sin^2 \theta - 5 \sin \theta + 1 = 0$$

or,

$$(3 \sin \theta - 1)(2 \sin \theta - 1) = 0$$

$$\sin \theta = 1/3 \text{ or } 1/2$$

$$\theta = 19° \text{ or } 30°.$$

The results should be checked.

2. Prove that for all values of θ (except $\pi/2$ and $3\pi/2$, for which the expressions are not defined)

$$\sec^4 \theta - \tan^4 \theta \equiv \tan^2 \theta + \sec^2 \theta.$$

3. Show that

$$\sec^2 u - \sin^2 u \equiv \tan^2 u + \cos^2 u,$$

for all values of the variable u except 90° and 270°, for which the expressions are not defined.

4. Find u, if

$$\tan u + \cot u = 2.$$

5. Find $\sec \theta$, if

$$2 \cos \theta + \sin \theta = 2.$$

6. Find the distance of the end of the diameter of

$$\rho = 8 \cos (\theta - 60°)$$

from the line OX.

7. If $\rho_1 = a \cos \theta$, and $\rho_2 = a \sin \theta$, find $\rho_1 - \rho_2$ when $\theta = 60°$ and $a = 5$.

8. Find the polar equation of the circle $x^2 + y^2 + 6x = 0$.

9. For what value of θ does $\rho = 3.55$, if $\rho = 2 \sin \theta + 3 \cos \theta$?
RESULT: $\theta = 23° \, 30'$ and $43° \, 30'$.

10. Prove that

$$\frac{\sin A}{1 - \cos A} \equiv \frac{1 + \cos A}{\sin A}.$$

11. Prove that $2 \cos^2 u - 1 \equiv \cos^4 u - \sin^4 u$.

12. Prove that

$$\sec u + \tan u \equiv \frac{1}{\sec u - \tan}.$$

13. Prove that $\sec^2 u + \csc^2 u \equiv \csc^2 u \sec^2 u$.

14. Show that $(\tan a + \cot a)^2 \equiv \sec^2 a \csc^2 a$.

15. Find $\sin \theta$ if $\csc \theta = \dfrac{\sqrt{a^2 + b^2}}{a}$.

16. A circular arc is 4.81 inches long. The radius is 12 inches. What angle is subtended by the arc at the center? Give result in radians and in degrees.

17. Certain lake shore lots are bounded by north and south lines 66 feet apart. How many feet of lake shore to each lot if the shore-line is straight and runs 77° 30′ E. of N.?

18. If $y = 2 \sin A + 3 \cos A - 3.55$, take A as 20°; as 23°; as 26°. Find in each case the value of y. From the values of y just found approximate the value of A for which y is just zero. This process is known as "**cut and try.**"

19. The line $y = (3/2)x$ is to coincide with the diameter of the circle:

$$\rho = 10 \cos (\theta - \alpha)$$

Find α.

20. The line $y = 2x$ is to coincide with the diameter of the circle:

$$\rho = 10 \sin (\theta + \alpha)$$

Find α.

21. To measure the width of the slide dovetail shown in Fig. 68, two carefully ground cylindrical gauges of standard dimensions are placed in the V's at A and B, as shown, and the distance X carefully

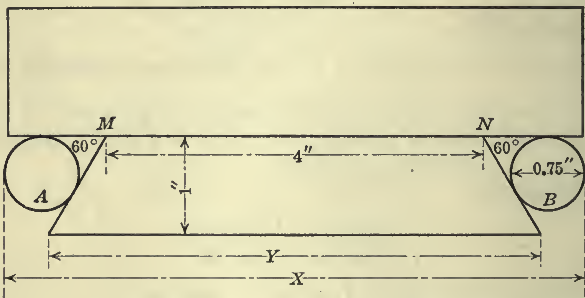


Fig. 68.—Diagram to Exercise 21.

taken with a micrometer. The angle of the dovetail is 60°. Find the reading of the micrometer when the piece is planed to the required dimension $MN = 4$ inches. Also find the distance Y. (Adapted from "Machinery," N. Y.)

22. Show that:

$$\rho = \sin \theta + \cos \theta$$

is a circle.

23 Draw the curve:

$$y = \sin x + \cos x.$$

24 Sketch

$$y = \frac{x}{2}$$

and

$$y = \sin x$$

and then

$$y = \frac{x}{2} + \sin x$$

and discuss.

CHAPTER IV

THE ELLIPSE AND HYPERBOLA

72. The Ellipse. If all ordinates of a circle be shortened by the same fractional amount of their length, the resulting curve is called an **ellipse.** For example, in Fig. 69, the middle points of the positive and negative ordinates of the circle were marked and a curve drawn through the points so selected. The result is the ellipse $ABA'B'A$.

If

$$x^2 + y^2 = a^2 \tag{1}$$

is the equation of a circle, then

$$x^2 + (my)^2 = a^2 \tag{2}$$

in which m is any constant > 1, is the equation of an ellipse; for substituting my for y divides all of the ordinates by m, by Theorem V on Loci, §27. The ellipse may also be looked upon as the orthographic projection of the circle. See §28.

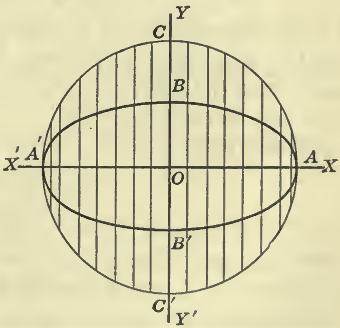


FIG. 69.—Definition of the Ellipse.

It is easy to show, as a consequence of the above, that the shadow cast on the floor by a circular disk held at any angle in the path of vertical rays of light is an ellipse.

The curve made by elongating by the same fractional amount of their length all of the abscissas or ordinates of a circle is also an ellipse, as the following considerations will show.

First let the ordinates of the circle (1) be shortened as before. The result is

$$x^2 + (my)^2 = a^2 \tag{2}$$

If the abscissas of the same given circle be multiplied by m to make another curve, the result is

$$\left(\frac{x}{m}\right)^2 + y^2 = a^2 \tag{3}$$

where m is supposed to be > 1 in both cases. If equation (3) be multiplied through by m^2 we get:

$$x^2 + (my)^2 = a^2m^2 \tag{4}$$

This shows that the second curve can be made by dividing by m all of the ordinates of a circle of radius ma. That is, (3) is an ellipse made from a circle of radius ma in the same manner that the ellipse (2) is made from a circle of radius a. Hence (3) is an ellipse whose dimensions are m-fold those of (2).

Thus an ellipse results if all of the ordinates or if all of the abscissas of a circle be multiplied or divided by any given constant m.

It is usual to write the multiplier m in the form a/b, so that equation (1) may be written:

$$x^2 + (ay/b)^2 = a^2$$

or:

$$\mathbf{x^2/a^2 + y^2/b^2 = 1} \tag{5}$$

which is the equation of the ellipse in a **symmetrical form.** Applying the principles of **§27,** the locus (5) may be thought of as made from the unit circle $x^2 + y^2 = 1$ by multiplying its abscissas by a and its ordinates by b.

When written:

$$y = \pm (b/a)\sqrt{a^2 - x^2} \tag{6}$$

$$y = \pm \sqrt{a^2 - x^2} \tag{7}$$

the ellipse and circle are placed in a form most useful for many purposes. It is easy to see that (6) states that its ordinates are the fractional amount b/a of those of the circle (7).

In Fig. 69 the points A and A' are called the **vertices** and the point O is called the **center** of the ellipse. The line AA' is called the **major axis** and the line BB' is called the **minor axis.** It is obvious that $AA' = 2a$, and from equation (5) or (6) it follows $BB' = 2b$.

The definition of the term *function* permits us to speak of y as a function of x, or of x as a function of y, in cases like equation (5) above; for when x is given, y is determined. To distinguish this from the case in which the equation is solved for y, as in (6), y, in the former case, is said to be an **implicit function** of x, and in the latter case y is said to be an **explicit function** of x.

If a circular cylinder be cut by a plane, the section of the cylinder is an ellipse. For select any diameter of a circular section of the cylinder as the x-axis. Let a plane be passed through this diameter making an angle α with the circular section. Then if ordinates (or chords perpendicular to the common x-axis) be drawn in each of the two planes, all ordinates of the section made by the cutting plane can be made from the ordinates of the circular section by multiplying them by sec α. Hence any plane section of a cylinder is an ellipse.

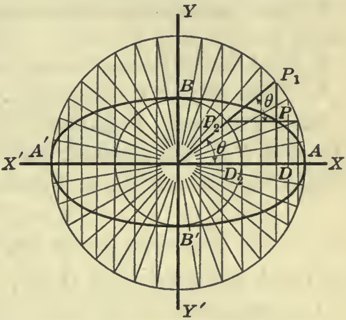


Fig. 70.—A Construction of the Ellipse.

73. To Draw the Ellipse. A method of drawing the ellipse is shown in Fig. 70. Draw concentric circles of radii a and b respectively, $a > b$. Draw any number of radii and from their intersections with the larger circle draw vertical lines, and from their intersections with the smaller circles draw horizontal lines. The points of intersection of the corresponding horizontal and vertical lines are points of the ellipse.

Proof. In the figure, let P be one of the points just described. Then:

$$P_2D_2 : P_1D = P_2O : P_1O$$

or, substituting PD for the equal P_2D_2

$$PD : P_1D = P_2O : P_1O$$

Now $OP_1 = a$ and $OP_2 = b$ and P_1D is the ordinate of the circle of radius a or is equal to $\sqrt{a^2 - x^2}$. Substituting these in the last proportion and solving for PD we obtain:

$$PD = y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

This is the equation of an ellipse. Hence the curve APB is an ellipse.

The two circles are called the **major** and **minor auxiliary circles.** The vectorial angle θ of P_1 is called the **eccentric angle** or the **eccentric anomaly** of the point P.

Exercises

1. Draw an ellipse whose semi-axes are 5 and 3, and write its equation.

2. From what circle can the ellipse $y = \pm \frac{1}{2}\sqrt{9 - x^2}$ be made by shortening of its ordinates?

3 Write the equation of the ellipse whose major axis is 7 and minor axis is 5.

4. Find the major and minor axes of the ellipse $x^2/7 + y^2/17 = 1$.

5. What curve is represented by the equation $x^2/9 + y^2/46 = 1$?

74. Parametric Equations of the Ellipse. From Fig. 70, OD and PD, the abscissa and ordinate of any point P of the ellipse, may be written as follows:

$$\begin{aligned} \mathbf{x} &= \mathbf{a} \cos \theta \\ \mathbf{y} &= \mathbf{b} \sin \theta \end{aligned} \tag{1}$$

for OD is the projection of $OP_1 = a$ through the angle θ and DP is the projection of $OP_2 = b$ through the angle $\pi/2 - \theta$. The pair of equations (1) is known as **the parametric equations of the ellipse.** The angle θ, in this use, is called the **parameter.** Writing (1) in the form:

$$\frac{x}{a} = \cos \theta$$

$$\frac{y}{b} = \sin \theta$$

squaring, and adding, we eliminate θ and obtain:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2}$$

the symmetrical equation of the ellipse.

If the abscissa and ordinate of any point of a curve are expressed in terms of a third variable, the pair of equations are called the **parametric equations** of the curve. Thus:

$$x = 4t$$
$$y = t + 1$$

are the parametric equations of a certain straight line. Its ordinary equation can be found by eliminating the parameter t between these equations.

From equations (1) we see that the ellipse might be defined as follows: Lay off distances on the X-axis proportional to $\cos\theta$, and distances on the Y-axis proportional to $\sin\theta$. Draw horizontal and vertical lines through the points of division, thus dividing the rectangle $2a$, $2b$ into a large number of small rectangles. Starting at the point $(a, 0)$ and drawing the diagonals of successive cornering rectangles, a line is obtained which approaches the ellipse as near as we please as the number of small rectangles is indefinitely increased. The student should draw this diagram. See Fig. 129.

Exercises

1. Draw the curve whose parametric equations are:

$$x = \cos\theta$$
$$y = \sin\theta.$$

2. Write the equation of the ellipse whose major and minor axes are 10 and 6, respectively.

3 Find the axes of the ellipse whose equation is:

$$y = \pm \tfrac{1}{2}\sqrt{36 - x^2}.$$

4. Write the parametric equations of the ellipse:

$$y = \pm \tfrac{2}{3}\sqrt{81 - x^2}.$$

5. Discuss the curve:

$$x = \pm \tfrac{1}{2}\sqrt{4 - y^2}.$$

6. Discuss the curves:

$$x^2 + 4y^2 = 1$$
$$4x^2 + y^2 = 1$$
$$(1/4)x^2 + y^2 = 1.$$

7. Write the Cartesian equation of the curves whose parametric equations are:

(a) $\begin{bmatrix} x = 2\cos\theta \\ y = \sin\theta \end{bmatrix}$ (b) $\begin{bmatrix} x = 6\cos\theta \\ y = 2\sin\theta \end{bmatrix}$ (c) $\begin{bmatrix} x = \sqrt{3}\cos\theta \\ y = \sqrt{2}\sin\theta. \end{bmatrix}$

8. What locus is represented by the parametric equations

$$x = 2t + 1$$
$$y = 3t + 5?$$

9. Show that

$$x = at$$
$$y = bt$$

is a line of slope b/a.

10. Write the equation of an ellipse whose major and minor axes are 6 and 4 respectively.

11. What curve is represented by the parametric equations:

$$x = 2 + 6 \cos \theta$$
$$y = 5 + 2 \sin \theta?$$

12. Show that the curve

$$x = 3 + 3 \cos \theta$$
$$y = 2 + 2 \sin \theta$$

is tangent to the coördinate axes.

13. The sunlight enters a dark room through a circular aperture of radius 8 inches, in a vertical window and strikes the floor at an angle of 60°. Find the dimensions and the equation of the boundary of the spot of light on the floor.

14. The ellipse

$$y = \pm \tfrac{2}{3}\sqrt{9 - x^2}$$

is the section of a circular cylinder. Find the angle α made by the cutting plane and the axis of the cylinder.

75.[1] **Other Methods of Constructing an Ellipse.** The following methods of constructing an ellipse of semi-axes a and b may be explained by the student from the brief outlines given:

1. Move any line whose length is $a + b$ (see Fig. 71) in such a manner that the ends A and B always lie on the X- and Y-axes, respectively. The point P describes an ellipse.

2. Mark on the edge of a straight ruler three points P, M, N, Fig. 72, such that $PM = b$ and $PN = a$. Then move the ruler keeping M and N always on AA' and BB' respectively. P describes an ellipse. The elliptic "trammel" of "ellipsograph" is constructed on this principle by use of adjustable pins on PMN and grooves on AA' and BB'.

3. Draw a semicircle of radius a about the center C, Fig. 73, and produce a radius to O such that $CTO = a + b$. From C draw

[1] This section may be omitted altogether or assigned as problems to various members of the class.

any number of lines to the tangent to the circle at T. From O draw lines meeting the tangent at the same points of TN. At the points where the lines from C cut the semicircle, draw parallels

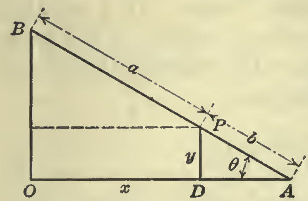


Fig. 71.—Ellipse Traced by a Point P of the Moving Line AB.

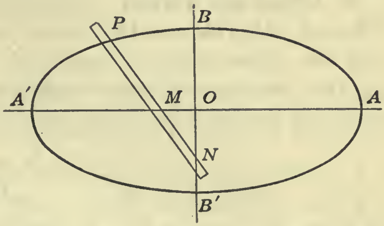


Fig. 72.—Theory of the Common "Ellipsograph" or Elliptic Trammel.

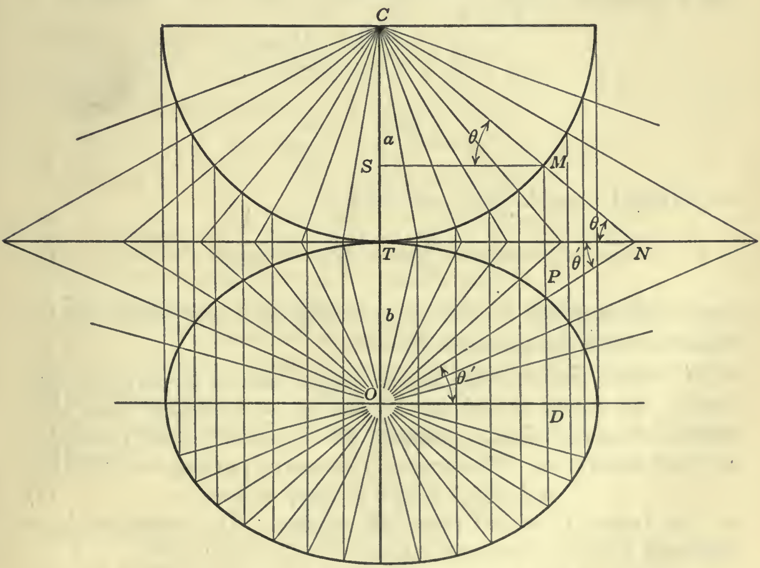


Fig. 73.—A Graphical Construction of an Ellipse.

to CT. The points of meeting of the latter with the lines radiating from O determine points on the ellipse.

To prove the above, note that $OD = a \cos \theta$, $PD = OD \tan \theta'$, also that $\tan \theta : \tan \theta' :: a : b$. Discuss the latter case when $b = a$ and also when $b > a$.

76. Origin at a Vertex. The equations of the ellipse (5) and (6) §**72** and (1) §**74** are the most useful forms. It is obvious that the ellipse may be translated to any position in the plane by the methods already explained. The ellipse with center moved to the point (h, k) has the equation:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{1}$$

Of special importance is the equation of the ellipse when the origin is taken at the left-hand vertex. This form is best obtained from equation (6), §**72,** by translating the curve the distance a in the x direction. Thus:

$$y = \pm \frac{b}{a}\sqrt{a^2 - (x-a)^2}$$

or,

$$y^2 = \frac{2b^2}{a}x - \frac{b^2}{a^2}x^2$$

or, letting l stand for the coefficient of x,

$$y^2 = lx - \frac{b^2}{a^2}x^2 = lx(1 - x/2a) \tag{2}$$

For small values of x, $x/2a$ is very small and the ellipse nearly coincides with the parabola $y^2 = lx$.

77. *Any equation of the second degree, lacking the term xy and having the terms containing x^2 and y^2 both present and with coefficients of like signs, represents an ellipse with axes parallel to the coördinate axes.* This is readily shown by putting the equation

$$ax^2 + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

in the form (1) of the preceding section. The procedure is as follows:

$$a\left(x^2 + 2\frac{g}{a}x\right) + b\left(y^2 + 2\frac{f}{b}y\right) = -c \tag{2}$$

$$a\left(x^2 + 2\frac{g}{a}x + \frac{g^2}{a^2}\right) + b\left(y^2 + 2\frac{f}{b}y + \frac{f^2}{b^2}\right) = \frac{g^2}{a} + \frac{f^2}{b} - c \tag{3}$$

Let M stand for the expression in the right-hand member of (3), then we get:

$$\frac{\left[x+\frac{g}{a}\right]^2}{\frac{M}{a}}+\frac{\left[y+\frac{f}{b}\right]^2}{\frac{M}{b}}=1 \tag{4}$$

This shows that (1) is an ellipse whose center is at the point $\left(-\frac{g}{a}, -\frac{f}{b}\right)$ and which is constructed from the circles whose centers are at the same point and whose radii are the square roots of the denominators in (4). The major axis is parallel to OX or OY according as a is less or greater than b. The cases when the locus is not real should be noted. Compare §**42.**

Illustration: Find the center and axes of the elllipse

$$x^2 + 4y^2 + 6x - 8y = 23$$

Write the equation in the form

$$x^2 + 6x + 4y^2 - 8y = 23$$

Complete the squares

$$x^2 + 6x + 9 + 4y^2 - 8y + 4 = 36$$

Rewriting $$(x + 3)^2 + 4(y - 1)^2 = 36$$

or $$(x + 3)^2/36 + (y - 1)^2/9 = 1$$

This is seen to be an ellipse whose center is at the point $(-3, 1)$ and whose semi-axes are $a = 6$ and $b = 3$.

The rotation of the ellipse through any angle about O as a center will be considered in another place. It should be noted, however, that the ellipse is turned through 90° by merely interchanging x and y.

78. Limiting Lines of an Ellipse. It is obvious from the equation

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

that $x = a$ and $x = -a$ are limiting lines beyond which the curve cannot extend; that is, x cannot exceed a in numerical value without y becoming imaginary. The same test may be applied to equations of the form:

$$x^2 + 4x + 9y^2 - 6y + 4 = 0$$

10

Solving for y in terms of x:

$$3y = 1 \pm \sqrt{1 - (x + 2)^2}$$

The values of y become imaginary when:

$$(x + 2)^2 > 1$$

or,

$$x + 2 > +1 \text{ or } < -1$$

or,

$$x > -1 \text{ or } < -3$$

These, then, are the limiting lines in the x direction. Finding the limiting lines in the y direction in the same way, the rectangle within which the ellipse must lie is determined.

In cases like the above the actual process of finding the limiting lines and the location of the center of the ellipse is best carried out by the method of §77.

Exercises

Find the lengths of the semi-axes and the coördinates of the center for the six following loci and translate the curves so that the terms in x and y disappear, by the method of §77.

1. $12x^2 - 48x + 3y^2 + 6y = 13.$
2. $y^2 - 8y + 4x^2 + 6 = 0.$
3. $x^2 - 6x + 4y^2 + 8y = 5.$
4. $x^2 + 9y^2 - 12x + 6y = 12.$
5. $4x^2 + y^2 - 12x + 2y - 2 = 0.$
6. $x^2 + 2y^2 - x - \sqrt{2}y = 1/2.$
7. Show that $x^2 + 4x + 9y^2 - 6y = 0$ passes through the origin.
8. Show that $x^2 - 4x + 4y^2 + 8y + 4 = 0$ is an ellipse.
9. Discuss the curves:

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$\frac{x^2}{9} + \frac{y^2}{4} = -1$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 0.$$

10. Discuss the following parabolas:

$$y = 2px^2$$
$$y = -2px^2$$
$$y = \frac{k}{h^2}x^2$$
$$y = -2px^2 + b.$$

What are the roots of the last function?

11. Write the symmetrical equation of the ellipse if its parametric equations are:

$$x = (3/2)\cos\theta$$
$$y = (2/3)\sin\theta.$$

12. Discuss the curve $y^2 = (18/5)x - (9/25)x^2$.

13. Compare the curves $y^2 = x - x^2$ and $y^2 = x$.

14. Find the center of the curve $y^2 = 2x(6 - x)$.

79. Graph of y = tan x. If this graph is to be constructed on a sheet of ordinary letter paper, $8\frac{1}{2}$ inches $\times$ 11 inches, it is desirable to proceed as follows: Draw at the left of the sheet of paper a semicircle of radius 1.15 . . . inches, (that is, of radius $= 18/5\pi$), so that the length of the arc of an angle of 10° or $\pi/18$ radians will be 1/5 inch. Take for the x-axis a radius COX prolonged, and take for the y-axis the tangent OY drawn through O, as in Fig. 74. Divide the semicircle into eighteen equal parts and draw radii through the points of division and prolong them to meet OY in points T_1, T_2, T_3, T_4, . . . Then on the y-axis there is laid off a scale YY' in which the distances OT_1, OT_2, . . . are proportional to the tangents of the angles OCS_1, OCS_2, . . .; for the tangents of these angles are OT_1/CO, OT_2/CO, . . . and CO is the unit of measure made use of throughout this diagram. Draw horizontal lines through the points of division on OY and vertical lines through the points of division on OX, thus dividing the plane into a large number of small rectangles. Starting at O, π, 2π, . . . $-\pi$, -2π, . . . and sketching the diagonals of consecutive cornering rectangles, the curve of tangents is approximated. Greater precision may be obtained by increasing as desired the number of divisions of the circle and the number of corresponding vertical and horizontal lines.

It is observed that the graph of the tangent is a series of similar branches, which are discontinuous for $x = \pi/2$, $-\pi/2$, $(3/2)\pi$,

$-(3/2)\pi$, . . . For these values of x the curve has vertical asymptotes, as shown at AB, $A'B'$, in Fig. 74.

If the number of corresponding vertical and horizontal lines be increased sufficiently, the slope of the diagonal of any rectangle gives a close approximation to the true slope of the curve at that point.

It has already been noted that all of the trigonometric functions are periodic functions of period 2π. It is seen in this case, however,

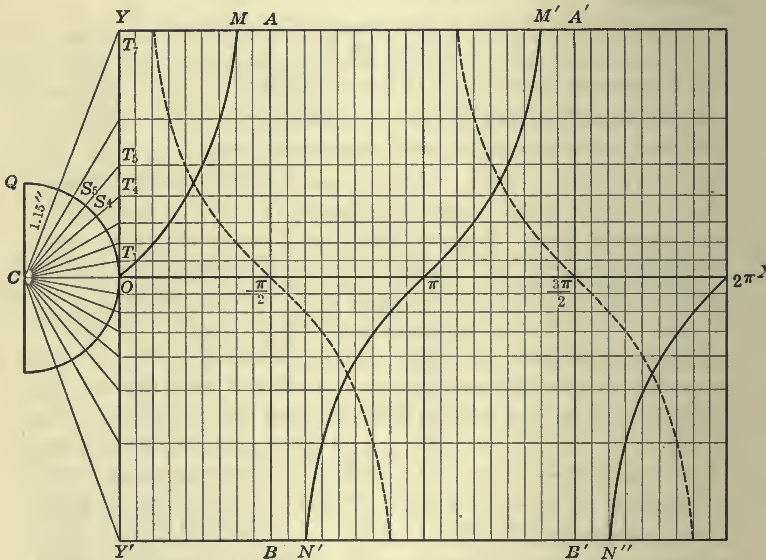


FIG. 74.—Graphical Construction of the Curve of Tangents $y = \tan x$. For lack of room only a few of the points S_1, S_2, . . . T_1, T_2,, . . . are lettered in the diagram. The dotted curve is $y = \cot x$.

that $\tan x$ has also the shorter period π; for the pattern MN, $M'N'$, $M''N''$, of Fig. 74 is repeated for each interval π of the variable x.

80. Ratio $(\sin x)/x$ and $(\tan x)/x$ for Small Values of x. Precisely as in the case of the locus of $y = \sin x$, the rectangles along, and on both sides of, the x-axis in the graph of $y = \tan x$, are nearly squares. In Fig. 74, the x-sides of these rectangles are

1 /5 inch, but the y-sides are slightly greater, since OT_1 is slightly greater than the arc OS_1 of the circle. To prove this, note that OT_1 is half of one side of a regular 18-sided polygon circumscribed about the circle; since the perimeter of this polygon is greater than the circumference of the circle, $OT_1 > OS_1$, for these magnitudes are 1 /36 of the perimeter and circumference, respectively, just named. Likewise in Fig. 59, $DS_1 < OS_1$, for DS_1 is one-half of the side of an 18-sided regular polygon inscribed in the circle and OS_1 is 1 /36 of the circumscribed circumference.
Hence:

$$\sin x < x < \tan x \tag{1}$$

or dividing by $\sin x$,

$$1 < \frac{x}{\sin x} < \sec x \tag{2}$$

Now as x approaches 0, the last term of this inequality approaches unity. Hence the second term, whose value always lies between the first and third term of the inequality, must approach the same value, 1. This fact is expressed in mathematics by the statement

$$\text{the limit of } \frac{\sin x}{x} = 1 \text{ as } x \text{ approaches } 0$$

or, in symbols:

$$\lim_{x \doteq 0} \frac{\sin x}{x} = 1 \tag{3}$$

Dividing (1) by $\tan x$,

$$\cos x < \frac{x}{\tan x} < 1 \tag{4}$$

Now as x approaches 0, the first term of this inequality approaches unity. Hence the second term, whose value always lies between the first and third term of the inequality, must approach the same value, 1. This fact is expressed by the statement

$$\text{the limit of } \frac{\tan x}{x} = 1 \text{ as } x \text{ approaches } 0$$

or, in symbols:

$$\lim_{x \doteq 0} \frac{\tan x}{x} = 1 \tag{5}$$

Equations (3) and (5) express very useful and important facts. Geometrically they state that the rectangles along the x-axis in

Figs. 59 and 74, approach more and more nearly *squares* as the number of intervals in the circle is increased. Each of the ratios in (2) approaches as near as we please to unity the smaller x is taken, but the limits of these ratios are unity *only when the angles are measured in radians.*

The word "limit" used above stands for the same concept that arises in elementary geometry. It may be formally defined as follows:

DEFINITION: A constant, a, is called the **limit** of a variable, t, if, as t runs through a sequence of numbers, the difference $(a - t)$ becomes, and remains, numerically smaller than any preassigned number.

81. Graph of cot x. In order to lay off a sequence of values of $\cot \theta$ on a scale, it is convenient to keep the denominator constant in the ratio (abscissa) / (ordinate) which defines the cotangent.

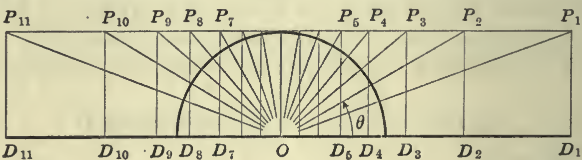


FIG. 75.—Construction of a Scale of Cotangents.

The denominator may also, for convenience, be taken equal to unity. Thus, in Fig. 75, the triangles of reference D_1OP_1, D_2OP_2, . . . for the various values of θ shown, have been drawn so that the ordinates P_1D_1, P_2D_2, . . . are equal. If the constant ordinate be also the unit of measure, then the sequence OD_1, OD_2, OD_3, . . . OD_7, OD_8, represents, in magnitude and sign, the cotangents of the various values of the argument θ. Using OD_1, OD_2, . . . as the successive ordinates and the circular measure of θ as the successive abscissas, the graph of $y = \cot x$ is drawn, as shown by the dotted curve in Fig. 74.

The sequence OD_1, OD_2, . . . Fig. 75 is exactly the same as the sequence OT_1, OT_2, . . . Fig. 74, but arranged in the reverse order. Hence, the graph of the cotangent and of the tangent are alike in general form, but one curve descends as the other ascends, so that the position, in the plane xy, of the branches of the curve are quite different. In fact, if the curve of the tangents be rotated about OY as axis and then translated to the right the distance $\pi/2$, the curves would become identical. Therefore, for all values of x:

$$\tan(\pi/2 - x) = \cot x \qquad (1)$$

This is a result previously known.

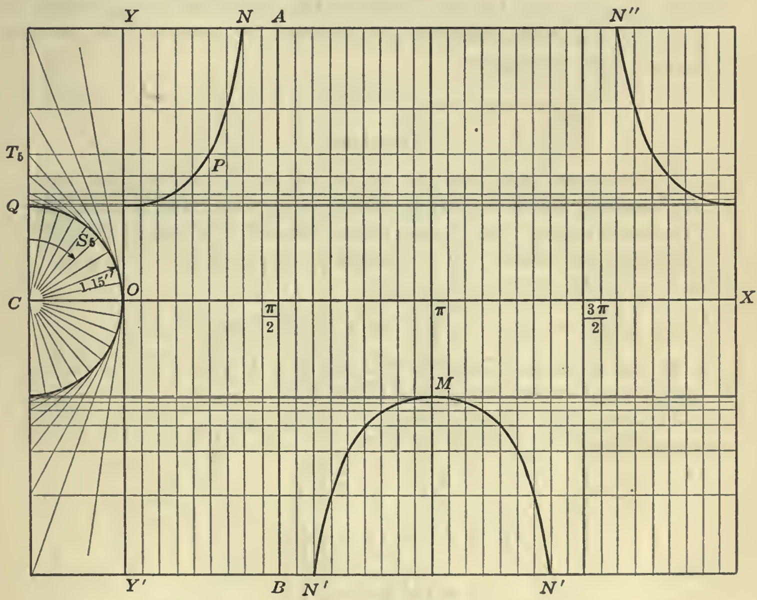


FIG. 76.—Graphical Construction of $y = \sec x$.

82. Graph of y = sec x. Since sec θ is the ratio of the radius divided by the abscissa of any point on the terminal side of the angle θ, it is desirable, in laying off a scale of a sequence of values of sec θ, to draw a series of triangles of reference with the abscissas in all cases the same, as shown in Fig. 76. In this figure the angles were laid off from CQ as initial line. Thus:

$$CT_5/CS_5 = \sec QCS_5$$

or, if CS_5 be unity, the distances like CT_5, laid off on CQ, are the secants of the angles laid off on the arc QS_5O or laid off on the axis OX.

The student may describe the manner in which the rectangles made by drawing horizontal lines through the points of division on CQ and the vertical lines drawn at equal intervals along OX, may be used to construct the curve. If the radius of the circle be 1.15 inches, what should be the length of $O\pi$ in inches?

The student may construct and discuss the locus of $y = \csc x$. Compare with the locus

$$y = \sec x$$

Exercises

1. Discuss from the diagrams, 59, 74, 76, the following statements:

Any number, however large or small, is the tangent of some angle.

The sine or cosine of any angle cannot exceed 1 in numerical value.

The secant or cosecant of any angle is always numerically greater than 1 (or at least equal to 1).

2. Show that $\sec\left(\frac{\pi}{2} - x\right) = \csc x$ for all values of x.

3. If $\tan\theta \sec\theta = 1$, show that $\sin\theta = \frac{1}{2}(\sqrt{5} - 1)$ and find θ by use of polar coördinate paper, Form $M3$.

4. Describe fully the following, locating nodes, troughs, crests, asymptotes, etc.:

$$y = \sin\left(x - \frac{\pi}{6}\right)$$

$$y = \cos\left(x + \frac{\pi}{6}\right)$$

$$y = \tan\left(x + \frac{\pi}{4}\right)$$

$$y = \tan\ (x + 1).$$

83. Increasing and Decreasing Functions. The meanings of these terms have been explained in §26. Applying these terms to the circular functions, we may say that $y = \sin x$, $y = \tan x$, $y = \sec x$ are increasing functions for $0 < x < \pi/2$. The co-functions, $y = \cos x$, $y = \cot x$, $y = \csc x$, are decreasing functions within the same interval.

Exercises

Discuss the following topics from a consideration of the graphs of the functions:

1. In which quadrants is the sine an increasing function of the angle? In which a decreasing function?

2. In which quadrants is the tangent an increasing, and in which a decreasing, function of its variable?

3. In which quadrants are the $\cos\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$, increasing and in which are they decreasing functions of θ?

4. Show that all the co-functions of angles of the first quadrant are decreasing functions.

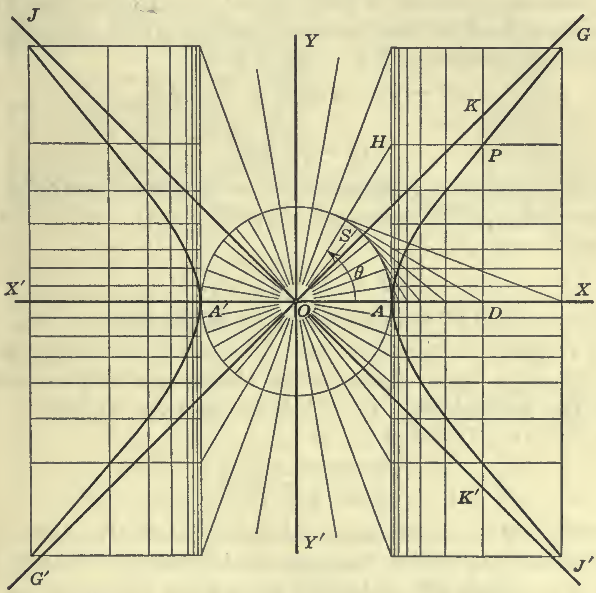


FIG. 77.—Construction of the Rectangular Hyperbola.

84. The Rectangular Hyperbola. We have seen that the circle is the locus of a point whose abscissa is $a\cos\theta$ and whose ordinate is $a\sin\theta$. The rectangular, or equilateral, hyperbola may be defined to be the locus of a point whose abscissa is $a\sec\theta$ and whose ordinate is $a\tan\theta$. To construct the curve, divide the X-axis proportionally to $\sec\theta$, and the Y-axis proportionally to $\tan\theta$, as shown in Fig. 77. The scale OX of this diagram may be taken from OY of Fig. 76, and the scale OY may be taken from OY of Fig.

74. The plane of xy may be divided into a large number of rectangles by passing lines through the points of division perpendicular to the scales and then, starting from A and A', sketching the diagonals of the successive cornering rectangles.

The parametric equations of the curve are, by definition:

$$\left.\begin{aligned} \mathbf{x} &= \mathbf{a}\sec\theta \\ \mathbf{y} &= \mathbf{a}\tan\theta \end{aligned}\right] \tag{1}$$

The Cartesian equation is easily found by squaring each of the equations and subtracting the second from the first, thus eliminating θ by the relation $\sec^2\theta - \tan^2\theta = 1$:

$$x^2 - y^2 = a^2(\sec^2\theta - \tan^2\theta)$$

or,

$$\mathbf{x}^2 - \mathbf{y}^2 = \mathbf{a}^2 \tag{2}$$

This is the Cartesian equation of the rectangular hyperbola.

The equation of the rectangular hyperbola may also be written in the useful form:

$$\mathbf{y} = \pm\sqrt{\mathbf{x}^2 - \mathbf{a}^2} \tag{3}$$

Compare (1) and (3) with the equations of the circle.

The rectangular hyperbola here defined will be shown, in §**86,** to be the curve $2xy = a^2$ rotated 45° clockwise about the origin.

85. The Asymptotes. Let $G'G$ be the line $y = x$, Fig. 77. The slope of OP[1] is PD/OD or y/x or

$$\frac{a\tan\theta}{a\sec\theta} = \sin\theta.$$

The value of θ corresponding to the point P is AOH. As the point P moves upward and to the right on the curve, the angle θ, or AOH, approaches 90° and $\sin\theta$ approaches unity. Hence the line OP approaches OG as a limit, and P approaches as near as we please to OG. The same reasoning applies to points moving out on the curve in the other quadrants. The lines GG' and JJ' are called **asymptotes** to the hyperbola.

86. The Curves $\mathbf{2xy = a^2}$ and $\mathbf{x^2 - y^2 = a^2}$. In Fig. 78, let the curve be the locus $2x_1y_1 = a^2$, referred to the axes $X_1'X_1$ and $Y_1'Y_1$. This curve has already been called the rectangular hyperbola. (See §**23.**) We desire to find the equation of the curve

[1] To avoid an excessive number of construction lines, OP is not shown in the figure.

referred to the axes $X_2X'_2$ and $Y_2Y'_2$. In the figure, y_1 is the sum of the projections of x_2 and y_2 on PD_1. The angle of projection is 45°, whose cosine is $\frac{1}{2}\sqrt{2}$. Hence,

$$y_1 = \tfrac{1}{2}\sqrt{2}(y_2 + x_2) \tag{1}$$

Likewise, x_1 is *the difference* in the projections through 45° of x_2 and y_2 on $X_1X'_1$. Or:

$$x_1 = \tfrac{1}{2}\sqrt{2}(x_2 - y_2) \tag{2}$$

Hence, multiplying the members of (1) and (2):

$$2x_1y_1 = x_2^2 - y_2^2 \tag{3}$$

Since by hypothesis $2x_1y_1 = a^2$, the equation of the curve referred to the axes X_2Y_2 is

$$x_2^2 - y_2^2 = a^2 \tag{4}$$

Thus, $2xy = a^2$ is the curve $x^2 - y^2 = a^2$ turned anti-clockwise through an angle of 45°.

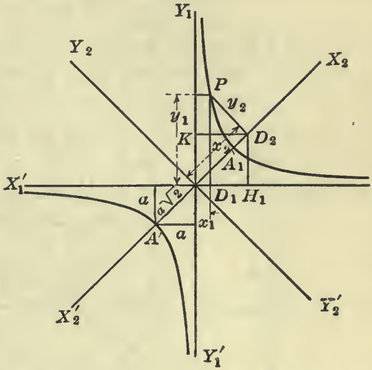


FIG. 78.—Comparison of $2xy = a^2$ and $x^2 - y^2 = a^2$.

By **§27**, the curve $2xy = a^2$ may be made from $xy = 1$ by multiplying both the abscissas and the ordinates by $a/\sqrt{2}$.

Are the curves $xy = 1$ and $x^2 - y^2 = 1$ of the same size?

87. Hyperbola of Semi-axes a and b. The curve whose abscissas are proportional to sec θ and whose ordinates are proportional to tan θ is called the **hyperbola.** Its parametric equations are, therefore:

$$\left.\begin{aligned} \mathbf{x} &= \mathbf{a} \sec \theta \\ \mathbf{y} &= \mathbf{b} \tan \theta \end{aligned}\right] \tag{1}$$

where a and b are constants.

To construct the curve, draw two concentric circles of radii a and b, respectively, as in Fig. 79. Divide both circumferences into the same number of convenient intervals. Lay off, on XOX', distances equal to a sec θ by drawing tangents at the points of division on the circumference of the a-circle; also lay off distances equal to $b \tan \theta$ on the vertical tangent to the b-circle by prolonging the radii of the latter through the points of division of the circumference. Draw horizontal and vertical lines through the points of division of MN and XX' respectively, dividing the plane into a large number of rectangles which are used exactly as in Fig. 77 for the construction of the curve.

In the above construction, there is no reason why the diameter of the b-circle may not exceed that of the a-circle.

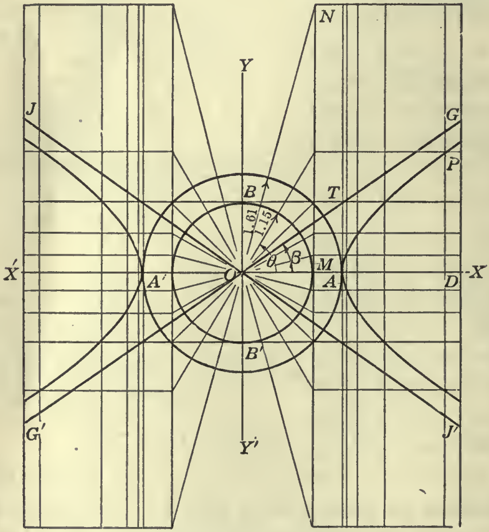


FIG. 79.—The Hyperbola $x^2/a^2 - y^2/b^2 = 1$.

Writing (1) in the form:

$$\left.\begin{aligned} \frac{x}{a} &= \sec \theta \\ \frac{y}{b} &= \tan \theta \end{aligned}\right]$$

and eliminating θ as before we obtain:

$$\frac{\mathbf{x}^2}{\mathbf{a}^2} - \frac{\mathbf{y}^2}{\mathbf{b}^2} = \mathbf{1} \qquad (2)$$

the Cartesian equation of the hyperbola. This is also called the **symmetrical equation** of the hyperbola.

The line $AA' = 2a$ is called the **transverse axis,** the line BB' is called the **conjugate axis,** the points A and A' are called the **vertices,** and the point O is called the **center** of the hyperbola. Let the line $G'OG$ be the line through the origin of slope b/a and let $J'OJ$ be the line of slope $-b/a$. The slope of the radius vector OP is:

$$\frac{PD}{OD} = \frac{y}{x} = \frac{b \tan \theta}{a \sec \theta} = \frac{b}{a} \sin \theta$$

The limit of this ratio as the point P moves out on the curve away from O is b/a; for θ approaches 90° as P moves outward, and hence $\sin \theta$ approaches 1. Hence, the line OP approaches in direction OG as a limit. Points moving along the curve away from O in the other quadrants likewise approach as near as we please to $G'G$ or $J'J$. The lines $G'G$ and $J'J$ are called the **asymptotes** of the hyperbola. The equations of these lines are

$$\mathbf{y} = \pm \frac{\mathbf{b}}{\mathbf{a}}\mathbf{x} \tag{3}$$

Solving the equation (2) for y, the equation of the hyperbola may be written in the useful form

$$\mathbf{y} = \pm \frac{\mathbf{b}}{\mathbf{a}}\sqrt{\mathbf{x}^2 - \mathbf{a}^2} \tag{4}$$

Compare this equation with the equation of the ellipse, (6) § **72.**

It is easy to show that the vertical distance PG of any point of the curve from the asymptote $G'G$ can be made as small as we please by moving P outward on the curve away from O.

Write the equation of the hyperbola in the form

$$y_1 = \frac{b}{a}\sqrt{x^2 - a^2} \tag{5}$$

and the equation of the asymptote GG' in the form

$$y_2 = \frac{b}{a}x \tag{6}$$

Then:

$$PG = y_2 - y_1 = \frac{b}{a}(x - \sqrt{x^2 - a^2}) \tag{7}$$

$$= \frac{b}{a}\frac{a^2}{x + \sqrt{x^2 - a^2}} \tag{8}$$

by multiplying both numerator and denominator in (7) by $x + \sqrt{x^2 - a^2}$. Now, as x increases in value without limit the right side of (8) approaches zero. Whence:

$$PG \doteq 0 \text{ as } x \doteq \infty$$

Exercises

1. Write the symmetrical equation of the hyperbola from the parametric equations $x = 5 \sec \theta$, $y = 3 \tan \theta$.

2. Find the Cartesian equation of the hyperbola from the relations $x = 7 \sec \theta$, $y = 10 \tan \theta$. Note that the graphical construction of the hyperbola holds if $b > a$.

3. What curve is represented by the equation

$$\frac{(x-3)^2}{25} - \frac{(y+2)^2}{16} = 1?$$

4. What curve is represented by the equation $y = \frac{1}{2}\sqrt{x^2 - a^2}$?

5. Write the equation of a hyperbola having the asymptotes $y = \pm (3/4)x$, and transverse axis $= 24$.

6. Show that the curves

$$x^2 + 6x - y^2 - 4y + 4 = 0$$

and

$$(x+3)^2 - (y+2)^2 = 1$$

are the same, and show that each is a hyperbola.

7. What curve is represented by the equations

$$x = h + a \sec \theta$$
$$y = k + b \tan \theta?$$

8. Discuss the curve $x^2 - 8x - 2y^2 - 12y = 0$.

88. Orthographic Projections. When the equation of the hyperbola is written in the useful form

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2} \tag{1}$$

it is seen that the hyperbola may be looked upon as generated from the equilateral hyperbola

$$y = \pm \sqrt{x^2 - a^2} \tag{2}$$

by multiplying all of its ordinates by b/a.

89. Conjugate Hyperbolas. Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{1}$$

Interchanging x and y in this equation gives, by Theorem III on Loci, §**24**, a new locus which is the reflection of (1) in the line $y = x$. The new equation may be written in the form

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1 \tag{2}$$

in which all signs have been changed after interchanging x and y. Since (2) is the same curve as (1) but in a new position, it is still a hyperbola; its vertices are located on the Y-axis instead of on the X-axis. The asymptotes of (1) have been found to be

$$y = \pm \frac{b}{a}x \tag{3}$$

Therefore the asymptotes of (2) may be found by reflecting (3) in the line $y = x$; hence they must be given by:

$$y = \pm \frac{a}{b}x \tag{4}$$

Now, if the constants a and b in equation (2) be interchanged giving thereby the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \tag{5}$$

then the shape of the hyperbola (2) will be changed but its position will be unaltered, that is, its vertices will still be located on the Y-axis. The asymptotes of (5) are found, of course, by interchanging a and b in (4), which gives an equation exactly like (3). Hence the hyperbola (5) has the same asymptotes as the original hyperbola (1). When a hyperbola with vertices on the Y-axis has the same asymptotes as a hyperbola with vertices on the X-axis, and of such size that the transverse axis of one hyperbola is the conjugate axis of the other, then the two hyperbolas are said to be **conjugate to each**

FIG. 80.—A Family of Conjugate Pairs of Hyperbolas with Common Asymptotes. (An interference pattern made from a glass plate under compression. From R. Strauble, "Ueber die Elasticitäts-zahlen und moduln des Glases." Wied. Ann. Bd. 68, 1899, p. 381.)

other. Thus (1) and (5) are two hyperbolas which are conjugate to each other. Obviously a hyperbola and its conjugate completely bound the space about the origin, except the cuts or lines represented by the common asymptotes.

Fig. 80 shows a family of pairs of conjugate hyperbolas.

Exercises

1. Sketch on the same pair of axes the four following hyperbolas and their asymptotes:

(1) $x^2 - y^2 = 25$ (3) $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$

(2) $x^2 - y^2 = -25$ (4) $\dfrac{x^2}{25} - \dfrac{y^2}{9} = -1.$

2. Find the axes of the hyperbola $y = \pm \frac{3}{4}\sqrt{x^2 - 64}.$

3. Compare the curves:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$$

4. Compare the curves:

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

and

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

5. Write the equation of the hyperbola conjugate to

$$y = \pm \tfrac{3}{4}\sqrt{x^2 - 64}.$$

6. Compare the graphs of:

$$y = \pm \tfrac{3}{4}\sqrt{x^2 - 64}$$
$$y = \pm \tfrac{3}{4}\sqrt{x^2 - 16}$$
$$y = \pm \tfrac{3}{4}\sqrt{x^2 - 4}$$
$$y = \pm \tfrac{3}{4}\sqrt{x^2 - 1}$$
$$y = \pm \tfrac{3}{4}\sqrt{x^2 - 1/16}$$
$$y = \pm \tfrac{3}{4}\sqrt{x^2 - 0}.$$

7. Show that $3x^2 - 4y^2 - 7x + 5y + 2 = 0$ is a hyperbola. Find the position of the center and of the vertices. The vertices locate the so-called "limiting lines" of the hyperbola.

8. Show that $x^2 - 4x - 4y^2 + 4y = 4$ is a hyperbola. Find the limiting lines and center.

9. Discuss the graphs:

$$x^2 - y^2 = 1$$

and

$$y^2 - x^2 = 1.$$

10. Discuss the graph $16x^2 - y^2 - 40x - 6y = 2$, and find the limiting lines.

11. In Fig. 77, show that $DS = PD$ and, hence, from the triangle DSO, $x^2 - y^2 = a^2$.

12. In Fig. 77, show that $PK = x - y$, $PK' = x + y$, and that the rectangle $PK \times PK'$ is constant for all positions of P and equal to the square on OA.

CHAPTER V

SINGLE AND SIMULTANEOUS EQUATIONS

90. The Rational Integral Function of x. The general form of a polynomial of the nth degree is:

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ .\ .\ .\ + a_{n-1}x + a_n$$

where the symbols, a_0, a_1, a_2, . . . , stand for any real constants whatsoever, positive or negative, integral or fractional, rational or irrational, and where n is any positive integer. The number of terms in the rational, integral function of the nth degree is $(n + 1)$.

91. The Remainder Theorem. *If a rational integral function of x be divided by $(x - r)$ the remainder which does not contain x is obtained by writing, in the given function, r in place of x:* This theorem means, for example that the remainder of the division:

$(x^3 - 6x^2 + 11x - 6) \div (x - 4)$ is $4^3 - 6(4)^2 + 11(4) - 6$ or 6

Also that the remainder of the division:

$$(x^3 - 6x^2 + 11x - 6) \div (x + 1)$$

is

$$(-1)^3 - 6(-1)^2 + 11(-1) - 6 = -24$$

The theorem enables one to write the remainder without actually performing the division.

To prove the theorem, let

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ .\ .\ .\ + a_{n-1}x + a_n \quad (1)$$

and: $f(r) = a_0r^n + a_1r^{n-1} + a_2r^{n-2} + \ .\ .\ .\ + a_{n-1}r + a_n \quad (2)$

then:

$$f(x) - f(r) = a_0(x^n - r^n) + a_1(x^{n-1} - r^{n-1}) + \ .\ .\ .\ + a_{n-1}(x - r) \quad (3)$$

The right side of this equation is made up of a series of terms containing differences of like powers of x and r, and, hence, by the well-known theorem in factoring,[1] each binomial term is exactly divisible by $(x - r)$. The quotient of the right side of (3) by

[1] See Appendix.

$(x - r)$ may be written out at length, but it is sufficient to abbreviate it by the symbol $Q(x)$ and write:

$$\frac{f(x) - f(r)}{x - r} = Q(x) \tag{4}$$

or:

$$\frac{f(x)}{x - r} = Q(x) + \frac{f(r)}{x - r} \tag{5}$$

Now if N be any dividend, D any divisor, and Q the quotient and R the remainder, then:

$$N/D = Q + R/D \tag{6}$$

This form applied to (5) shows that $f(r)$ is the remainder when $f(x)$ is divided by $(x - r)$. Thus the Remainder Theorem is established.

92. The Factor Theorem. *If a rational integral function of x becomes zero when r is written in the place of x, $(x - r)$ is a factor of the function:* This means, for example, that if 3 be substituted for x in the function $x^3 - 6x^2 + 11x - 6$ and the result $3^3 - 6(3)^2 + 11(3) - 6 = 0$, then $(x - 3)$ is a factor of $x^3 - 6x^2 + 11x - 6$.

This theorem is but a corollary to the remainder theorem. For if the substitution $x = r$ renders the function zero, the remainder when the function is divided by $(x - r)$ is zero, and the theorem is established.

The value r of the variable x that causes the function to take on the value zero has already been named a **root** or a **zero** of the function. The factor theorem may, therefore, be stated in the form: *A rational integral function of the variable x is exactly divisible by $(x - r)$ where r is any root of the function.*

The familiar method of solving a quadratic equation by factoring is nothing but a special case of the present theorem. Thus if:

$$x^2 - 5x + 6 = 0$$

then:

$$(x - 2)(x - 3) = 0$$

and the roots are $x = 2$ and $x = 3$. The numbers 2 and 3 are such that when substituted in $x^2 - 5x + 6$ the expression is zero; and the factors of the expression are $x - 2$ and $x - 3$ by the factor theorem.

Exercises

1. Tabulating the cubic polynomial $x^3 - 6x^2 + 11x - 6$, we obtain:

x	-3	-2	-1	-0	1	1.5	2	2.5	3	4
$f(x)$,	-120,	-60,	-24,	-6,	0,	$+0.375$,	0,	-0.375,	0,	6

What is the remainder when the function is divided by $x - 4$? By $x + 2$? By $x + 3$? By $x - 1.5$? By $x - 3$?

Name three factors of the above function.

2. Find the remainder when $x^4 - 5x^3 + 12x^2 + 4x - 8$ is divided by $x - 2$.

3. Show by the remainder theorem that $x^n + a^n$ is divisible by $x + a$ when n is an odd integer, but that the remainder is $2a^n$ when n is an even integer.

4. Without actual division, show that $x^4 - 4x^2 - 7x - 24$ is divisible by $x - 3$.

5. Show that $a^4 + a^2 - ab^3 - b^2$ is divisible by $a - b$.

6. Show that $(b - c)(b + c)^2 + (c - a)(c + a)^2 + (a - b)(a + b)^2$ is divisible by $(b - c)(c - a)(a - b)$.

7. Show that $(x + 1)^2(x - 2) - 4(x - 1)(x - 5) + 4$ is divisible by $x - 1$.

8. Show that $(b - c)^3 + (c - a)^3 + (a - b)^3$ is divisible by $(b - c)(c - a)(a - b)$.

9. Show that $6x^5 - 3x^4 - 5x^3 + 5x^2 - 2x - 3$ is divisible by $x + 1$.

93. It follows at once from the factor theorem that it is possible to set up an equation with any roots desired; for example, if we desire an equation with the roots 1, 2, 3 we have merely to write:

$$(x - 1)(x - 2)(x - 3) = 0 \tag{1}$$

Forming the product:

$$x^3 - 6x^2 + 11x - 6 = 0$$

or transposing the terms in any manner, as:

$$x^3 + 11x = 6x^2 + 6$$

in no way essentially modifies the equation. If, however, the equation (1) be multiplied through by any function of x, the number of roots of the equation may be increased. Thus, multiplying (1) by $(x + 2)$ introduces a new root $x = -2$. Likewise, dividing equation (1) through by the factor $(x - 2)$, leaves an equation:

$$(x - 1)(x - 3) = 0 \tag{2}$$

which lacks the root $x = 2$.

By the principles or axioms of algebra, an *equation remains true* if we unite the same number to both sides by addition or subtraction; or if we multiply or divide both members by the same number, *not zero;* or if like powers or roots of both members be taken. But we have given sufficient illustrations to show that these operations *may affect the number of roots of the equation.* This is obvious enough in the cases already cited. Sometimes, however, the operation that removes or introduces a root is so natural and its effect is so disguised that the student is not apt to take due account of its effect. Thus, the roots of:

$$3(x - 5) = x(x - 5) + x^2 - 25 \tag{3}$$

are -1 and 5, for either of these when substituted for x will satisfy the equation. Dividing the equation through by $x - 5$, the resulting equation is:

$$3 = x + x + 5 \tag{4}$$

This equation is not satisfied by $x = 5$. One root has disappeared in the transformation. This is easy to keep account of if (3) be given in the form:

$$(x - 5)(x + 1) = 0, \tag{5}$$

but the fact that a factor has been removed may be overlooked when the equation is written in the form first given.

A very important effect upon the roots of an equation results from squaring both members. The student must always take proper account of the effect of this common operation. To illustrate, take the equation:

$$x + 5 = 1 - 2x \tag{6}$$

It is satisfied only by the value $x = -4/3$. Now, by squaring both sides of the equation, we obtain:

$$x^2 + 10x + 25 = 1 - 4x + 4x^2 \tag{7}$$

which is satisfied by either $x = 6$ or $x = -4/3$. Here, obviously, an extraneous solution has been introduced by the operation of squaring both members.

It is easy to show that squaring both members of an equation is equivalent to multiplying both sides by the sum of the left and right members. Thus, let any equation be represented by:

$$L(x) = R(x) \tag{8}$$

in which $L(x)$ represents the given function of x that stands on the left side of the equation and $R(x)$ represents the given function of x that stands on the right side of the equation.

Squaring both sides:

$$[L(x)]^2 = [R(x)]^2 \tag{9}$$

Transposing:

$$[L(x)]^2 - [R(x)]^2 = 0 \tag{10}$$

or factoring;

$$[L(x) + R(x)]\,[L(x) - R(x)] = 0 \tag{11}$$

But (8) may be written:

$$L(x) - R(x) = 0 \tag{12}$$

Thus, by squaring the members of the equation the factor $L(x) + R(x)$ has been introduced.

The sum of the left and right members of (6), above, is $6 - x$. Hence, squaring both sides of (6) is equivalent to the introduction of this factor, or, the operation introduces the root 6, as already noted.

As another example, suppose that it is required to solve:

$$\sin \alpha \cos \alpha = 1/4 \tag{13}$$

for $\alpha < 90°$. Substituting for $\cos \alpha$:

$$\sin \alpha \sqrt{1 - \sin^2 \alpha} = 1/4 \tag{14}$$

squaring:

$$\sin^2 \alpha\,(1 - \sin^2 \alpha) = 1/16 \tag{15}$$

completing the square:

$$\sin^4 \alpha - \sin^2 \alpha + 1/4 = 3/16 \tag{16}$$

Hence:

$$\begin{aligned} \sin \alpha &= \pm \sqrt{1/2 \pm (1/4)\sqrt{3}} \\ &= \pm\, 0.9659 \text{ or } \pm\, 0.2588 \end{aligned} \tag{17}$$

Only the positive values satisfy (13); the negative values were introduced in squaring (14). If, however, *the restriction* $\alpha < 90°$ *be removed, so that the radical in* (14) *must be written with the double sign, then no new solutions are introduced by squaring.*

94. Legitimate and Questionable Transformations. If one equation is derived from another by an operation which has no effect one way or another on the solution, it is spoken of as a **legitimate transformation**; if the operation does have an effect upon the final result, it is called a **questionable transformation,** meaning thereby that the effect of the operation requires examination.

In performing operations on the members of equations, the effect on the solution must be noted, and proper allowance made in the result. It cannot be too strongly emphasized that *the test for any solution of an equation is that it satisfy the original equation.* "No matter how elaborate or ingenious the process by which the solution has been obtained, if it do not stand this test it is no solution; and, on the other hand, no matter how simply obtained, provided it do stand this test, it is a solution."[1]

Among the common operations that have no effect on the solution are multiplication or division by known numbers, or addition or subtraction of like terms to both members; none of these introduce factors containing the unknown number. Taking the square root of both numbers is legitimate *if the double sign be given to the radical.* Clearing of fractions is legitimate if it be done so as not to introduce a new factor. If the fractions are not in their lowest terms, or if the equation be multiplied through by an expression having more factors than the least common multiple of the denominators, new solutions may appear, for extra factors are probably thereby introduced. Hence, in clearing of fractions the multiplier should be the least common denominator and the fractions should be in their lowest terms. This, however, does not constitute a *sufficient condition,* therefore *the only certainty lies in checking all results.*

Exercises

Suggestions: It is important to know that any equation of the form

$$ax^{2n} + bx^n + c = 0$$

can be solved as a quadratic by finding the two values of x^n. Frequently equations of this type appear in the form

$$dx^n + ex^{-n} = f$$

[1] Chrystal's Algebra.

Likewise any equation of the form

$$af(x) + b\sqrt{f(x)} + c = 0$$

can be solved as a quadratic by finding the two values of $\sqrt{f(x)}$ and then solving the two equations resulting from putting $\sqrt{f(x)}$ equal to each of them. One of these usually gives extraneous solutions.

These two types occur in the exercises given below.

Since operations which introduce extraneous solutions are often used in solving equations, the only sure test for the solution of any equation is to check the results by substituting them in the original equation.

Take account of all questionable operations in solving the following equations:

1. $\dfrac{3x}{x-3} = \dfrac{6}{x+3} + \dfrac{9}{x-3}.$

2. $(x^2 + 5x + 6)/(x - 3) + 4x - 7 = -15.$

3. $3(x - 5)(x - 1)(x - 2) = (x - 5)(x + 2)(x + 3).$

NOTE: Divide by $(x - 5)$, but take account of its effect.

4. $x^2/a + ax = x^2/b + bx.$

5. $ax(cx - 3b) = 5a(3b - cx).$

6. $x^2 - n^2 = n - x.$

7. $(x - 4)^3 + (x - 5)^3 = 31[(x - 4)^2 - (x - 5)^2].$ Divide by $(x - 4) + (x - 5)$ or $2x - 9$.

8. $\dfrac{x^2 - 3x}{x^2 - 1} + 2 + \dfrac{1}{x - 1} = 0.$ If the fractions be added, multiplication is unnecessary. There is only one root.

9. $x = 7 - \sqrt{x^2 - 7}.$

10. $\sqrt{x + 20} - \sqrt{x - 1} - 3 = 0.$

11. $\sqrt{15/4 + x} = 3/2 + \sqrt{x}.$

12. $20x/\sqrt{10x - 9} - \sqrt{10x - 9} = 18/\sqrt{10x - 9} + 9.$

13. $\dfrac{\sqrt{x} + \sqrt{x - 3}}{\sqrt{x} - \sqrt{x - 3}} = \dfrac{3}{x - 3}.$ Consider as a proportion and take by composition and division.

14. $x^{2/3} + 5/2 = (13/4)x^{1/3}.$

15. $\sqrt[4]{x^3} - 2\sqrt{x} + x = 0.$ Divide by $\sqrt{x}$.

16. $2\sqrt{x^2 - 5x + 2} - x^2 + 8x = 3x - 6.$ Call $x^2 - 5x + 2 = u^2$.

17. $4x^2 - 4x + 20\sqrt{2x^2 - 5x + 6} = 6x + 66.$

18. $x^{-2} - 2x^{-1} = 8.$

19. $x^{2/11} - 5x^{1/11} + 4 = 0.$

20. $110x^{-4} + 1 = 21x^{-2}.$

21. $\sqrt{x} + 4x^{-1/2} = 5.$

22. $8x^{3/2} - 8x^{-3/2} = 63.$

23. $(x - a)^n - 3(x - a)^{-n} = 2.$

24. $2x^{1/3} - 3x^{2/3} + x = 0.$

95. Intersection of Loci. Any pair of values of x and y that satisfies an equation containing x and y locates some point on the graph of that equation. Consequently, any set of values of x and y that satisfies both equations of a system of two equations containing x and y, must locate some point common to the graphs of the two equations. In other words, the coördinates of a point of intersection of two graphs is a solution of the equations of the graphs considered as simultaneous equations.

To find the values of x and y that satisfy two equations, we solve them as simultaneous equations. Hence, to find the points of intersection of two loci we must solve the equations of the two curves. There will be a pair of values or a solution for each point of intersection.

Thus, the intersection of the lines $y = 3x - 2$ and $y = x/2 + 3$ is the point (2, 4) and $x = 2$, $y = 4$, is the solution of the simultaneous equations.

To find the points of intersection of the circle $x^2 + y^2 = 25$ and the straight line $x + y = 9$ we solve the equations by the usual method, as follows:

$$\left.\begin{aligned} x^2 + y^2 &= 25 \qquad (1)\\ x + y &= 7 \qquad (2)\end{aligned}\right\}$$

The graphs are a straight line and a circle, as shown in (1), Fig. 81. Squaring the second equation, the system becomes:

$$\left.\begin{aligned} x^2 + y^2 &= 25 \qquad (3)\\ x^2 + 2xy + y^2 &= 49 \qquad (4)\end{aligned}\right\}$$

The second equation represents the *two* straight lines shown in Fig. 81, (2). The effect of squaring has been to introduce two extraneous solutions corresponding to the points P_3 and P_4.

Multiplying (3) by 2 and subtracting (4) from it, the last pair of equations becomes:

$$\left.\begin{aligned} x^2 - 2xy + y^2 &= 1 \\ x^2 + 2xy + y^2 &= 49 \end{aligned}\right\} \qquad \begin{aligned}(6)\\(7)\end{aligned}$$

which gives the four straight lines of Fig. 81, (4). Taking the square root of each member, but discarding the equation $x + y = -7$, because it corresponds to the extraneous solutions introduced by the questionable operation, we have:

$$\left.\begin{aligned} x - y &= \pm 1 \\ x + y &= 7 \end{aligned}\right\} \qquad \begin{aligned}(8)\\(9)\end{aligned}$$

By addition and subtraction we obtain the results:

$$\left.\begin{aligned} x &= 3 \\ y &= 4 \end{aligned}\right\} \qquad (10)$$

$$\left.\begin{aligned} x &= 4 \\ y &= 3 \end{aligned}\right\} \qquad (11)$$

represented by the intersections of the lines parallel to the axes shown in Fig. 81, (5).

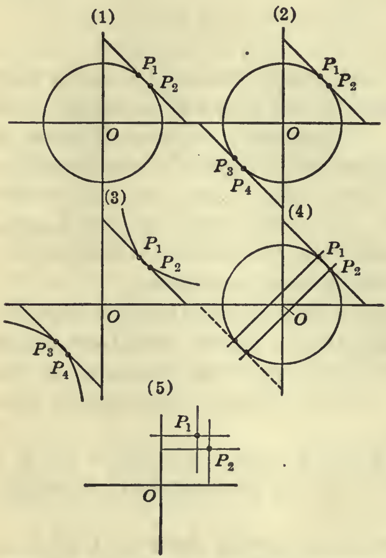


FIG. 81.—Graphic Representation of the Steps in the Solution of a Certain set of Simultaneous Equations.

This is a good illustration of the graphical changes that take place during the solution of simultaneous equations of the second degree. The ordinary algebraic solution consists, geometrically, in the successive replacement of loci by others of an entirely different kind, but all passing through the points of intersection (as P_1, P_2, Fig. 81) of the original loci.

Exercises

1. Find the points of intersection of the circle and parabola:

$$x^2 + y^2 = 5$$
$$y^2 = 4x.$$

Note that of the *two lines parallel to the y-axis, given by the equation* $x^2 + 4x - 5 = 0$, one does not cut the circle: $x^2 + y^2 = 5$.

2. Find the points of intersection of $x^2 + y^2 = 5$ and the hyperbola $x^2 - y^2 = 3$.

3. Solve, by graphical means only, to two decimal places:

$$y = x^2 + x - 1$$
$$xy = 1.$$

4. Solve in like manner:

$$x^2 + y^2 = 16$$
$$x^2 - 2xy + y^2 = 9.$$

Reason out what each equation represents before attempting to graph.

5. Solve in like manner:

$$x^2 + y^2 + x + y = 7$$
$$2x^2 + 2y^2 - 4x + 4y = 8.$$

These loci should be graphed without tabulating numerical values of the variables.

6. Solve graphically:

$$u^2 + v^2 = 9$$
$$u^2 - v^2 = 4.$$

NOTE. Draw the lines $x + y = 9$, and $x - y = 4$. The values of x and y determined by the intersection of these lines are the values of u^2 and v^2 respectively, from which u and v can be computed.

7. Solve the system:

$$x^2 + y^2 = 10$$
$$x^2/16 + y^2/9 = 1.$$

96. Quadratic Systems.[1] Any linear-quadratic system of simultaneous equations, such as:

$$y = mx + k$$
$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

can always be solved analytically; for y may readily be eliminated by substituting from the first equation into the second. A system of two quadratic equations may, however, lead, after elimination, to an equation of the third or fourth degree; and, hence, such equations cannot, in general, be solved until the solutions of the cubic and biquadratic equations have been explained.

[1] A large part of the remainder of this chapter can be omitted if the students have had a good course in algebra in the secondary school.

A single illustration will show that an equation of the fourth degree may result from the elimination of an unknown number between two quadratics. Thus, let:

$$x^2 - y = 5$$
$$x^2 + xy = 10$$

From the first, $y = x^2 - 5$. Substituting this value of y in the second equation, and performing the indicated operations, we obtain:

$$x^4 + x^3 - 5x + 10 = 0.$$

While, in general, a bi-quadratic equation results from the process of elimination from two quadratic equations, there are special cases of some importance in which the resulting equation is either a quadratic equation or a higher equation in the quadratic form. Two of these cases are:

(1) Systems in which the terms containing the unknown numbers are homogeneous; that is, systems in which the terms containing the unknown numbers are all of the *second degree* with respect to the unknown numbers, such, for example, as:

$$x^2 - 2xy = 5$$
$$3x^2 - 10y^2 = 35$$

(2) Systems in which both equations are *symmetrical;* that is, such that interchanging x and y in every term does not alter the equations; for example:

$$x^2 + y^2 - x - y = 78$$
$$xy + x + y = 39$$

97. Unknown Terms Homogeneous. The following work illustrates the reasoning that will lead to a solution when applied to any quadratic system all of whose terms containing x and y are of the second degree. Let the system be:

$$x^2 - xy = 2$$
$$2x^2 + y^2 = 9 \qquad (1)$$

Divide each through by x^2 (or y^2), then:

$$1 - (y/x) = 2/x^2$$
$$2 + (y/x)^2 = 9/x^2 \qquad (2)$$

Since the left members were homogeneous, dividing by x^2 renders them functions of the ratio (y/x) alone; call this ratio m. Then equations (2) contain only the unknown numbers m and x^2. The latter is readily eliminated by subtraction, leaving a quadratic for the determination of m. When m is known, substituting in (2) determines x, and the relation $y = mx$ determines the corresponding values of y.

The above illustrates the principles on which the solution is based. In practice, it is usual to substitute $y = mx$ at once, and then eliminate x^2 by comparison; thus, from the substitution $y = mx$ in (1), we obtain:

$$\begin{aligned} x^2 - mx^2 &= 2 \\ 2x^2 + m^2x^2 &= 9 \end{aligned} \tag{3}$$

Thence:

$$\begin{aligned} x^2 &= 2/(1 - m) \\ x^2 &= 9/(2 + m^2) \end{aligned} \tag{4}$$

Whence:

$$2/(1 - m) = 9/(2 + m^2) \tag{5}$$

or:

$$2m^2 + 9m = 5 \tag{6}$$

Factoring:

$$(2m - 1)(m + 5) = 0 \tag{7}$$

whence:

$$m = 1/2 \text{ or } -5 \tag{8}$$

Hence:

$$\begin{aligned} x &= \pm 2 \text{ or } \pm (1/3)\sqrt{3} \\ y &= \pm 1 \text{ or } \mp (5/3)\sqrt{3} \end{aligned} \tag{9}$$

These solutions should be written as corresponding pairs of values as follows:

$$\begin{array}{llll} x = 2 & x = -2 & x = (1/3)\sqrt{3} & x = -(1/3)\sqrt{3} \\ y = 1 & y = -1 & y = -(5/3)\sqrt{3} & y = (5/3)\sqrt{3} \end{array}$$

This system can readily be solved without the use of the mx substitution by merely solving the first equation for y and substituting in the second.

Graphically (see Fig. 82), the above problem is equivalent to finding the intersections of the curves:

$$\begin{aligned} x(x - y) &= 2 \\ (\sqrt{2}\,x)^2 + y^2 &= 9 \end{aligned}$$

The first is a curve with the two asymptotes $x = 0$ and $x - y$

$= 0$. As a matter of fact, the curve is a hyperbola, although proof that such is the case cannot be given until the method of rotating any curve about the origin has been explained. The second curve is obviously an ellipse generated from a circle of radius 3 by shortening the abscissas in the ratio $\sqrt{2}:1$. The two curves intersect at the points:

$x = 2$	-2	$0.557 \ldots$	$-0.557 \ldots$
$y = 1$	-1	$-2.887 \ldots$	$+2.887 \ldots$

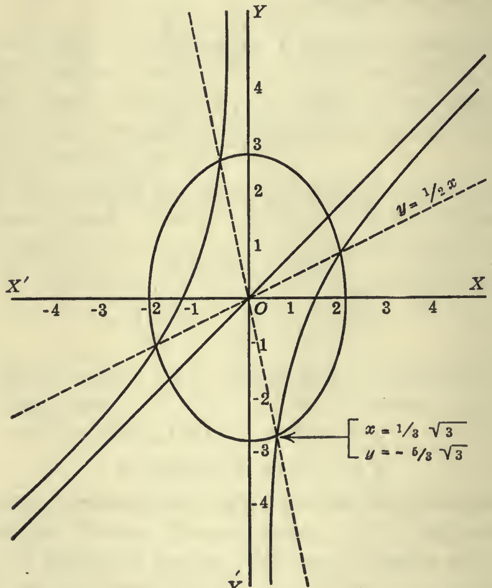


FIG. 82.—Solutions of a Set of Simultaneous Quadratics given graphically by the coördinates of the points of Intersection of the Ellipse and Hyperbola.

The auxiliary lines, $y = \frac{1}{2}x$ and $y = -5x$, made use of in the solution are shown by the dotted lines.

98. Symmetrical Systems. Simultaneous quadratics of this type are always readily solved analytically by seeking for the values of the binomials $x + y$ and $x - y$. The ingenuity of the student will usually show many short cuts or special expedients adapted to the particular problem. The following worked examples point out some of the more common artifices used.

1. Solve

$$x + y = 6 \qquad (1)$$

$$xy = 5 \qquad (2)$$

Squaring (1)

$$x^2 + 2xy + y^2 = 36 \qquad (3)$$

Subtracting four times (2) from (3):

$$x^2 - 2xy + y^2 = 16$$

whence:

$$x - y = \pm 4$$

But from (1):

$$x + y = 6$$

Therefore:

$$x = 5 \qquad x = 1$$
$$y = 1 \qquad y = 5$$

2. Solve

$$x^2 + y^2 = 34 \qquad (1)$$

$$xy = 15 \qquad (2)$$

Adding two times (2) to (1):

$$x^2 + 2xy + y^2 = 64 \qquad (3)$$

Subtracting two times (2) from (1):

$$x^2 - 2xy + y^2 = 4 \qquad (4)$$

Whence, from (3) and (4):

$$x + y = \pm 8$$
$$x - y = \pm 2$$

Therefore:

$$x = 5 \qquad x = 3 \qquad x = -5 \qquad x = -3$$
$$y = 3 \qquad y = 5 \qquad y = -3 \qquad y = -5$$

The hyperbola and circle represented by (1) and (2) should be drawn by the student.

3.

$$x^3 + y^3 = 72 \qquad (1)$$

$$x + y = 6 \qquad (2)$$

Cubing (2):

$$x^3 + 3x^2y + 3xy^2 + y^3 = 216 \qquad (3)$$

Subtracting (1) and dividing by 3:

$$xy(x + y) = 48 \tag{4}$$

whence, since

$$x + y = 6$$

we have

$$xy = 8 \tag{5}$$

From (2) and (5) proceed as in example 1, and find:

$$x = 4 \qquad x = 2$$
$$y = 2 \qquad y = 4$$

Otherwise, divide (1) by (2) and proceed by the usual method.

4. Solve

$$x^2 + xy = (7/3)(x + y) \tag{1}$$
$$y^2 + xy = (11/3)(x + y) \tag{2}$$

adding (1) and (2):

$$(x + y)^2 - 6(x + y) = 0 \tag{3}$$

whence:

$$x + y = 0 \text{ or } 6 \tag{4}$$

Now, because $x + y$ is a factor of both members of (1) and (2), the original equations are satisfied by the unlimited number of pairs of values of x and y whose sum is zero, namely, the coördinates of all points on the line $x + y = 0$.

Dividing (1) by (2), we get:

$$x/y = 7/11$$

This, and the line $x + y = 6$, from (4), give the solution:

$$x = 7/3$$
$$y = 11/3$$

Graphically, the equation (1) is the two straight lines:

$$(x - 7/3)(x + y) = 0$$

Equation (2) is the two straight lines:

$$(y - 11/3)(x + y) = 0$$

These loci intersect in the point (7/3, 11/3) and also intersect everywhere on the line $x + y = 0$.

Exercises

1. Show that:

$$x^2 + y^2 = 25$$
$$x + y = 1$$

has a solution, but that there is no real solution of the system:

$$x^2 + y^2 = 25$$
$$x + y = 11.$$

2. Do the curves:

$$x^2 + y^2 = 25$$
$$xy = 100, \text{ intersect?}$$

Do the curves:

$$x^2 + y^2 = 25$$
$$xy = 12, \text{ intersect?}$$

3. Solve:

$$(x^2 + y^2)(x + y) = 272$$
$$x^2 + y^2 + x + y = 42.$$

NOTE: Call $x^2 + y^2 = u$, and $x + y = v$.

4. Show that there are four real solutions to:

$$x^2 + y^2 - 12 = x + y$$
$$xy + 8 = 2(x + y).$$

5. Solve:

$$x^2 + y^2 + x + y = 18$$
$$xy = 6.$$

99. Graphical Solution of the Cubic Equation. The roots of a cubic $x^3 + ax^2 + bx + c = 0$ (where a, b, and c are given known numbers) may be determined graphically as explained in **§39**, or we may proceed as follows: The next highest term in the cubic may be removed by the substitution $x = x_1 - a/3$, as may readily be shown by trial. Hence, it is merely necessary to consider cubic equations of the form:

$$x^3 + ax + b = 0 \tag{1}$$

Consider the system of equations:

$$\left.\begin{aligned} y &= x^3 \\ y &= -ax - b \end{aligned}\right\} \tag{2}$$

Graphically, the curves consist of the cubic parabola (Fig. 83) and a straight line. The intersections of the two graphs give the solutions of the system. Eliminating y by subtraction, we obtain

$$x^2 + ax + b = 0$$

which shows that the values of x that satisfy the system (2) give the roots of equation (1). Hence (1) may be solved by means of the graph of (2). In this graph the cubic parabola $y = x^3$ is the same for all cubics; hence if the cubic parabola be once drawn accurately to scale, then all cubic equations can be solved by properly drawing the appropriate straight line, or by properly laying a straight edge across the graph of the cubic parabola.

In drawing the graph of the cubic parabola, it is desirable to use, for the y-scale, one-tenth of the unit used for the x-scale, so as to bring a greater range of values for y upon an ordinary sheet of coördinate paper. The cubic parabola graphed to this scale is shown in Fig. 83. The diagram gives the solution of $x^3 - x - 1 = 0$. The graphs $y = x^3$ and $y = x + 1$ are seen to intersect at $x = 1.32$. This, then, should be one root of the cubic correct to two decimal places. The line $y = x + 1$ cuts the cubic parabola in but one point, which shows that there is but one real root of the cubic. To obtain the imaginary roots, divide $x^3 - x - 1$ by $x - 1.32$. The result of the division, retaining but two places of decimals in the coefficients, is:

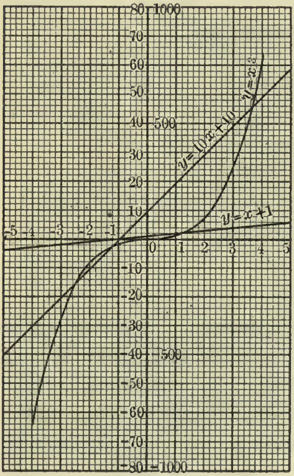


Fig. 83.—A Graphical Scheme for the Solution of Cubic Equations.

$$x^2 + 1.32x + 0.7424 \tag{3}$$

Putting this equal to zero and solving by completing the square, we find:

$$x = -0.66 \pm \sqrt{-0.3068}$$

or:

$$x = -0.66 \pm 0.55\sqrt{-1} \tag{4}$$

in which, of course, the coefficients are not correct to more than two places.

The equation:

$$x^3 - 10x - 10 = 0 \tag{5}$$

illustrates a case in which the cubic has three real roots. The straight line $y = 10x + 10$ cuts the cubic parabola (see Fig. 83) at $x = -1.2$, $x = -2.4$, and $x = 3.6$. These, then, are the approximate roots. The product:

$$(x + 1.2)(x + 2.4)(x - 3.6) = x^3 - 10.08x - 10.37$$

should give the original equation (5). This result checks the work to about two decimal places.

It is obvious that a similar process will apply to any equation of the form

$$x^n + ax + b = 0$$

The x-scale of Fig. 83 extends only from -5 to $+5$. The same diagram may, however, be used for any range of values by suitably changing the unit of measure on the two scales; thus, the divisions of the x-scale may be marked with numbers 5-fold the present numbers, in which case the numbers on the y-scale must be marked with numbers 125 times as great as the present numbers. These results are shown by the auxiliary numbers attached to the y-scale in Fig. 83.[1]

Exercises

Solve graphically the following equations checking each result separately.

1. $x^3 - 4x + 10 = 0.$

2. $x^3 - 12x - 8 = 0.$

3. $x^3 + x - 3 = 0.$

4. $x^3 - 15x - 5 = 0.$

5. $x^3 - 3x + 1 = 0.$

6. $x^3 - 4x - 2 = 0.$

7. $2 \sin \theta + 3 \cos \theta = 1.5.$

Note: Construct on polar paper the circles $\rho = 2 \sin \theta$ and $\rho = 3 \cos \theta$.

8. $2x + \sin x = 0.6.$

Note: Find the intersection of $y = \sin x$ and the line $y = -2x + 0.6$. If 1.15 inches is the amplitude of $y = \sin x$, then 1.15 must be the unit of measure used for the construction of the line $y = -2x + 0.6$.

9. $x^3 + x + 1 + 1/x = 0.$

10. Show that $x^3 + ax + b = 0$ can have but one real root if $a > 0$.

11. (*a*) Show that the graph of $y = x^3 + bx$ is symmetrical with respect to the origin. (See §**37**, equation (1).)

[1] For other graphical methods of solution of equaitons, see Runge's "Graphical Methods," Columbia University Press, 1912.

(*b*) Show that the graph of $y = x^3 + bx + c$ is symmetrical with respect to the point $(0, c)$.

(*c*) If the substitution $x = x_1 - a/3$ removes the term ax^2 from the equation $y = x^3 + ax^2 + bx + c$, show that the graph of this last equation must be symmetrical with respect to some point.

12. On polar paper, draw a curve showing the variation of local or mean solar time with the longitude of points on the earth's surface.

If it be *noon* by both standard and mean solar (local) time at Greenwich, longitude 0°, construct a graph on polar paper showing standard time at all other longitudes, if the longitude of a point be represented by the vectorial angle on polar paper and if *time* relative to Greenwich be represented on the radius vector using 1 cm. = 2 hours, and also if it be assumed that the changes of standard time take place exactly at 15° intervals beginning at $7\frac{1}{2}$° west longitude.

If it be noon at Greenwich, write an equation which will express the local time of any point in terms of the longitude of the point. Does the expression hold for points having negative longitude? Does this function possess a discontinuity?

Can a similar expression be written giving the standard time at any point in terms of the longitude of the point?

If t be standard time and θ longitude, and if the functional relation by expressed by f, so that:

$$t = f(\theta)$$

is f a continuous or a discontinuous function? Is the function f defined for $\theta = 15°, 30°, 45°$, etc., and why?

In actual practice, how is the function f given?

100. Method of Successive Approximations. The graphic method of solving numerical equations, combined with the method explained below, is the only method which is universally applicable. It therefore possesses a practical importance exceeding that of any other method. An example will illustrate the method.

Suppose that it is required to find to four decimal places one root of $x^3 - x - 1 = 0$. See §99 and Fig. 83. The graphic method gives $x = 1.32$. This is the first approximation. A second approximation is found as follows: Build the table of values for $y = x^3 - x - 1$

x	y	
1.32	− .0200	
1.33	+ .0226	
0.01	.0426	Differences.

Now reason as follows: The actual root lies between 1.32 and 1.33, and the zero value of y corresponds to it. This zero is 200/426 of the way between the two values of y; hence if the curve be nearly straight between $x = 1.32$, and $x = 1.33$, the desired value of x is approximately 200/426 of the way between 1.32 and 1.33 or it is $x = 1.324694$. This value is probably correct to the fourth decimal place.

To find a third approximation we build another table of values:

x	y	
1.3247	− .0000766	
1.3248	+ .0003499	
0.0001	.0004265	Differences.

Reasoning as before, we get $x = 1.324718$ which is very likely true to the last decimal place.

The above method is applicable to an equation like exercise 8 above. In fact it is the only numerical method that is applicable in such cases.

CHAPTER VI

PERMUTATIONS AND COMBINATIONS; THE BINOMIAL THEOREM

101. Fudamental Principle. *If one thing can be done in n different ways and another thing can be done in r different ways, then both things can be done together, or in succession, in $n \times r$ different ways.* This simple theorem is fundamental to the work of this chapter. To illustrate, if there be 3 ways of going from Madison to Chicago and 7 ways of going from Chicago to New York, then there are 21 ways of going from Madison to New York.

To prove the general theorem, note that if there be *only one* way of doing the first thing, *that* way could be associated with each of the r ways of doing the second thing, making r ways of doing both. That is, for each way of doing the first, there are r ways of doing both things; hence, for n ways of doing the first there are $n \times r$ ways of doing both.

Illustrations: A penny may fall in 2 ways; a common die may fall in 6 ways; the two may fall together in 12 ways.

In a society, any one of 9 seniors is eligible for president and any one of 14 juniors is eligible for vice-president. The number of tickets possible is, therefore, 9×14 or 126.

I can purchase a present at any one of 4 shops. I can give it away to any one of 7 people. I can, therefore, purchase and give it away in any one of 28 different ways.

A product of two factors is to be made by selecting the first factor from the numbers a, b, c, and then selecting the second factor from the numbers x, y, z, u, v. The number of possible products is, therefore, 15.

If a first thing can be done in n different ways, a second in r different ways, and a third in s different ways, the three things can be done in $n \times r \times s$ different ways. This follows at once from the fundamental principle, since we may regard the first two things as constituting a single thing that can be done in nr

ways, and then associate it with the third, making $nr \times s$ ways of doing the *two* things, consisting of the first two *and* the third.

In the same way, if one thing can be done in n different ways, a second in r different ways, a third in s, a fourth in t, etc., then all can be done together in $n \times r \times s \times t \ldots$ different ways.

Thus, n different presents can be given to x men and a women in $(x + a)^n$ different ways. For the first of the n presents can be given away in $(x + a)$ different ways, the second can be given away in $(x + a)$ different ways, and the third in $(x + a)$ different ways and so on. Hence, the number of possible ways of giving away the n presents to $(x + a)$ men and women is:

$$(x + a)(x + a)(x + a) \ldots \text{ to } n \text{ factors, or } (x + a)^n$$

102. Definitions. Every distinct order in which objects may be placed in a line or row is called a **permutation** or an **arrangement.** Every distinct selection of objects that can be made, irrespective of the order in which they are placed, is called a **combination** or **group.**

Thus, if we take the letters a, b, c, two at a time, there are six arrangements, namely: *ab, ac, ba, bc, ca, cb,* but there are only three groups, namely: *ab, ac, bc.*

If we take the three letters *all* at a time, there are six arrangements possible, namely: *abc, acb, bca, bac, cab, cba,* but there is only *one* group, namely: *abc.*

Permutations and combinations are both results of mode of selection. Permutations are selections made with the understanding that two selections are considered as different *even though they differ in arrangement only;* combinations are selections made with the understanding that two selections are not considered as different, *if they differ in arrangement only.*

In the following work, products of the natural numbers like

$$1 \times 2 \times 3; \quad 1 \times 2 \times 3 \times 4 \times 5; \quad \text{etc.}$$

are of frequent occurrence. These products are abbreviated by the symbols $3!$ $5!$ and read **"factorial three," "factorial five"** respectively.

103. Formula for the Number of Permutations of n Different Things Taken All at a Time. We are required to find how many possible ways there are of arranging n different things in a line.

Lay out a row of n blank spaces, so that each may receive one of these objects, thus:

$$|\underline{\ 1\ }|\quad|\underline{\ 2\ }|\quad|\underline{\ 3\ }|\quad|\underline{\ 4\ }|\quad|\underline{\ 5\ }|\quad\ldots\;|\underline{\ n\ }|$$

In the first space we may place any one of the n objects; therefore, that space may be occupied in n different ways. The second space, after one object has been placed in the first space, may be occupied in $(n-1)$ different ways; hence, by the fundamental principle, the two spaces may be occupied in $n(n-1)$ different ways. In like manner, the third space may be occupied in $(n-2)$ different ways, and, by the same principle, the first three spaces may be occupied in $n(n-1)(n-2)$ different ways, and so on. The next to the last space can be occupied in but two different ways, since there are but two objects left, and the last space can be occupied in but one way by placing therein the last remaining object. Hence, the total number of different ways of occupying the n spaces in the row with the n objects is the product:

$$n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$$

or,

$$n!$$

If we use the symbol P_n to stand for the number of permutations of n things taken all at a time, then we write:

$$\mathbf{P}_n = \mathbf{n}! \tag{1}$$

104. Formula for the Number of Permutations of n Things Taken r at a Time. We are required to find how many possible ways there are of arranging a row consisting of r different things, when we may select the r things from a larger group of n different things.

For convenience in reasoning, lay out a row of r blank spaces, so that each of the spaces may receive one of the objects, thus:

$$|\underline{\ 1\ }|\quad|\underline{\ 2\ }|\quad|\underline{\ 3\ }|\quad\ldots\;|\underline{\ r-1\ }|\quad|\underline{\ r\ }|$$

In the first space of the row, we may place any one of the n objects; therefore, that space may be occupied in n different ways. The second space, after one object has been placed in the first space, may be occupied in $(n-1)$ different ways; hence, by the fundamental principle, the two spaces may be occupied in $n(n-1)$ different ways. In like manner, the third space may be occupied in $(n-2)$ different ways, and hence, the first three may be occupied in $n(n-1)(n-2)$ different ways, and so on. The last or rth space can be occupied in as many different ways as there are objects left. When an object is about to be selected for the rth space, there have been used $(r-1)$ objects (one for each of the $(r-1)$ spaces already occupied). Since there were n objects to begin with, the number of objects left is $n-(r-1)$ or $n-r+1$, which is the number of different ways in which the last space in the row may be occupied. Hence, the formula:

$$\mathbf{P}_{n,r} = \mathbf{n(n-1)(n-2)\ \ldots\ (n-r+1)} \qquad (1)$$

in which $P_{n,r}$ stands for the number of permutations of n things taken r at a time.

The formula, by multiplication and division by $\underline{|n-r}$, becomes:

$$\frac{n(n-1)\ \ldots\ (n-r+1)(n-r)(n-r-1)\ \ldots\ 3\cdot 2\cdot 1}{(n-r)(n-r-1)\ \ldots\ 3\cdot 2\cdot 1}$$

$$\mathbf{P}_{n,r} = \frac{\mathrm{n!}}{(\mathrm{n-r})!} \qquad (2)$$

This formula is more compact than the form (1) above, but the fraction is not in its lowest terms.

Formula (1) is easily remembered by the fact that there are just r factors beginning with n and decreasing by one. Thus we have:

$$P_{10,7} = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4$$

Exercises

1. How many permutations can be made of six things taken all at a time?

2. How many different numbers can be made with the five digits 1, 2, 3, 4, 5, using each digit once and only once to form each number?

3. The number of permutations of four things taken all at a time bears what ratio to the number of permutations of seven things taken all at a time?

4. How many arrangements can be made of eight things taken three at a time?

5. How many arrangements can be made of eight things taken five at a time?

6. How many four-figure numbers can be formed with the ten digits 0, 1, 2, . . . 9 without repeating any digit in any number?

7. How many different ways may the letters of the word *algebra* be written, using all of the letters?

8. How many different signals can be made with seven different flags, by hoisting them one above another five at a time?

9. How many different signals can be made with seven different flags, by hoisting them one above another *any number* at a time?

10. How many different arrangements can be made of nine ball players, supposing only two of them can catch and one pitch?

105. Formula for the number of combinations or groups of n different things taken r at a time.

It is obvious that the number of combinations or groups consisting of r objects each that can be selected from n objects, is less than the number of permutations of the same objects taken r at a time, for each combination or group when selected can be arranged in a large number of ways. In fact, since there are r objects in the group, each group can be arranged in exactly $r!$ different ways. Hence, for each group of r objects, selected from n objects, there exists $r!$ permutations of r objects each. Therefore, the number of permutations of n things, taken r at a time, is $r!$ times the number of combinations of n objects taken r at a time. Calling the unknown number of combinations x, we have:

$$xr! = P_{n,r} = \frac{n!}{(n-r)!}$$

or, solving for x:

$$x = \frac{n!}{r!(n-r)!}$$

This is the number of combinations of n objects taken r at a time, and may be symbolized:

$$\mathbf{C}_{n,r} = \frac{\mathbf{n!}}{\mathbf{r!(n-r)!}} \tag{1}$$

This fraction will always reduce to a whole number. It may be written in the useful form:

$$C_{n,r} = \frac{n(n-1)(n-2)\ .\ .\ .\ (n-r+1)}{1\times 2\times 3\quad .\ .\ .\quad r} \tag{2}$$

It is easily remembered in this form, for it has r factors in both the numerator and the denominator. Thus for the number of combinations of ten things taken four at a time we have four factors in the numerator and denominator, and

$$C_{10,4} = \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4}.$$

Exercises

1. How many different products of three each can be made with the five numbers a, b, c, d, e, provided each combination of three factors gives a different product.

2. How many products can be made from twelve different numbers, by taking eight numbers to form each product?

3. How many products can be made from twelve different numbers, by taking four numbers to form each product?

4. How many different hands of thirteen cards each can be held at a game of whist?

5. In how many ways can seven people sit at a round table?

6. In how many ways can a child be named, supposing that there are 400 different Christian names, without giving it more than three names?

7. In how many ways can a committee of three be appointed from six Germans, four Frenchmen, and seven Americans provided each nationality is represented?

8. There are five straight lines in a plane, no two of which are parallel; how many intersections are there?

9. There are five points in a plane, no three of which are collinear; how many lines result from joining each point to every other point?

10. In a plane there are n straight lines, no two of which are parallel; how many intersections are there?

11. In a plane there are n points, no three of which are collinear; how many straight lines do they determine?

12. In a plane there are n points, no three of which are collinear, except r, which are all in the same straight line; find the number of straight lines whch result from joining them.

13. A Yale lock contains five tumblers (cut pins), each capable of being placed in ten distinct positions. At a certain arrangement of the tumblers, the lock is open. How many locks of this kind can be made so that no two shall have the same key?

14. In how many ways can seven beads of different colors be strung so as to form a bracelet?

15. How many different sums of money can be formed from a dime, a quarter, a half dollar, a dollar, a quarter eagle, a half eagle, and an eagle?

106.* The Arithmetical Triangle. In deriving by actual multiplication, as below, any power of a binomial $x + a$ from the preceding power, it is easy to see that any coefficient in the new power is the sum of the coefficient of the corresponding term in the multiplicand and the coefficient preceding it in the multiplicand. Thus:

$$\begin{array}{r}
x^3 + 3ax^2 + 3a^2x + a^3 \qquad\quad \\
x + a \qquad\quad \\
\hline
x^4 + 3ax^3 + 3a^2x^2 + a^3x \qquad\quad \\
ax^3 + 3a^2x^2 + 3a^3x + a^4 \\
\hline
x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4
\end{array}$$

or, erasing coefficients, we have:

$$\begin{array}{r}
1 + 3 + 3 + 1 \qquad \\
1 + 1 \qquad \\
\hline
1 + 3 + 3 + 1 \qquad \\
1 + 3 + 3 + 1 \\
\hline
1 + 4 + 6 + 4 + 1
\end{array}$$

from which the law of formation of the coefficients 1, 4, 6, . . . is evident. Hence, writing down the coefficients of the powers of $x + a$ in order, we have:

Powers	Coefficients										
	1	2	3	4	5	6	7	8	9	10	11
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

In this triangle, each number is the sum of the number above it and the number to the left of the latter. Thus 84 in the 9th line equals $56 + 28$, etc. The triangle of numbers was used previous to the time of Isaac Newton for finding the coefficients of any desired power of a binomial. At that time it was little suspected that the coefficients of any power could be made without first obtaining the ocefficients of the preceding power. Isaac Newton, while an undergraduate at Cambridge, showed that the coefficients of any power could be found without knowing the coefficients of the preceding power; in fact, he showed that the coefficients of any power n of a binomial were functions of the exponent n.

The above triangle of numbers is known as the **arithmetical triangle** or as **Pascal's triangle.**

107. Distributive Law of Multiplication. The demonstration of the binomial theorem may be based upon the following law of multiplication: *The product of any number of polynomials is the aggregate of all the possible partial products which can be made by taking one term and only one from each of the polynomials.* This statement is merely a definition of what is meant by the product of two or more polynomials. (See appendix.) Thus:

$$(x + a)(y + b)(z + c) =$$
$$xyz + ayz + bxz + cxy + abz + bcx + cay + abc$$

Each of the eight partial products contains a letter from each parenthesis, and never two from the same parenthesis. The number of terms is the number of different ways in which a letter can be selected from each of the three parentheses. In the present case this is, by **§101,** $2 \times 2 \times 2 = 8$.

108. Binomial Formula. It is required to write out the value of $(x + a)^n$, where x and a stand for any two numbers and n is a positive integer. That is, we must consider the product of the n parentheses:

$$(x + a)(x + a)(x + a) \quad . \quad . \quad . \quad (x + a)$$

by the distributive law stated above.

First. Take an x from each of the parentheses to form *one* of the partial products. This gives the term x^n of the product.

Second. Take an a from the first parenthesis with an x from each of the other $(n - 1)$ parentheses. This gives ax^{n-1} as another partial product. But if we take a from the second parenthesis and an x from each of the other $(n - 1)$ parentheses, we get ax^{n-1} as another partial product. Likewise by taking a from *any* of the parentheses and an x from each of the other $(n - 1)$ parentheses, we shall obtain ax^{n-1} as a partial product. Hence, the final product contains n terms like ax^{n-1}, or nax^{n-1} *is a part of the product.*

Third. We may obtain a partial product like a^2x^{n-2} by taking an a from *any two* of the parentheses, together with the x's from each of the other $(n - 2)$ parentheses. Hence, there are as many partial products like a^2x^{n-2} as there are ways of selecting two a's from n parentheses; that is, as many ways as there are groups or combinations of n things taken two at a time, or:

$$\frac{n(n-1)}{1 \cdot 2}$$

Hence, $\dfrac{n(n-1)}{1 \cdot 2}\, a^n x^{n-2}$ is another part of the product.

Fourth. We may obtain a partial product like a^3x^{n-3} by taking an a from any three of the parentheses together with the x's from each of the other $(n - 3)$ parentheses. Hence, there are as many partial products like a^3x^{n-3} as there are ways of selecting three a's from n parentheses, that is, as many ways as there are combinations of n things taken three at a time, or $\dfrac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$.

Hence, $\dfrac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\, a^3x^{n-3}$ is another part of the product.

In general, we may obtain a partial product like a^rx^{n-r} (where r is an integer $< n$) by taking an a from *any* r of the parentheses together with the x's from each of the other $(n - r)$ parentheses. Hence, there are as many partial products like a^rx^{n-r} as there are ways of selecting r a's from n parentheses; that is, as many ways as there are combinations of n things taken r at a time, or $\dfrac{n!}{r!\,(n-r)!}$. Hence, $\dfrac{n!}{r!\,(n-r)!}\, a^rx^{n-r}$ stands for any term in general in the product $(x + a)^n$.

Finally, we may obtain one partial product like a^n by taking an a from each of the parentheses. Hence, a^n is the last term in the product.

Thus we have shown that:

$$(x + a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1\cdot 2} a^2x^{n-2} + \ . \ . \ .$$
$$+ \frac{n!}{r!\,(n-r)!} a^rx^{n-r} + \ . \ . \ . + a^n$$

This is the binomial formula of Isaac Newton. The right side is called the expansion or development of the power of the binomial.

It is obvious that the expansion of $(x - a)^n$ will differ from the above only in the signs of the alternate terms containing the odd powers of a, which, of course, will have the negative sign.

109. Binomial Theorem. The binomial expansion is a *series*, that is, each term may be derived from the preceding term by a definite law. This law is made up of two parts which may be stated as follows:

(1) *Law of Exponents. In any power of a binomial, $x + a$, the exponent of x commences in the first term with the exponent of the required power, and in the following terms continually decreases by unity. The exponent of a commences with* 1 *in the second term and continually increases by unity.*

(2) *Law of Coefficients. The coefficient in the first term is* 1, *that in the second term is the exponent of the power; and if the coefficient in any term be multiplied by the exponent of x in that term and divided by the exponent of a, increased by* 1, *it will give the coefficient in the succeeding term.*

Exercises

1. Expand $(u + 3y)^5$. Here $x = u$ and $a = 3y$. By the formula we get:

$$u^5 + 5u^4(3y) + 10u^3(3y)^2 + 10u^2(3y)^3 + 5u(3y)^4 + (3y)^5$$

Performing the indicated operations, we obtain:

$$u^5 + 15u^4y + 90u^3y^2 + 270u^2y^3 + 405uy^4 + 243y^5$$

Expand each of the following by the binomial formula:

2. $(r^2 - 2)^4$.

3. $(3b - 1/2)^5$.

4. $(c + x)^6$.

5. $(2x^2 - x)^6$.

6. $(1 - a)^8$.

7. $(-x + 2a)^7$.

8. $(1/2 + x)^5$.

9. $(b^2 - c^2)^5$.

10. $(3a + 1/2)^6$.

11. $(5d - 3y)^5$.

12. $(3x^3 - 1)^4$.

13. $(\sqrt{a} + x)^6$.

14. $(x^{3/2} + x^{2/3})^6$.

15. $(a^{-2} - b^{1/2})^4$.

16. $(\sqrt{ab} - \sqrt[3]{ab})^6$.

17. $(a + [x + y])^3$.

18. $(a + b - y)^3$.

19. $(x^2 + 2ax + a^2)^3$.

110. Binomial Theorem for Fractional and Negative Exponents. It is proved in the Calculus that:

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

is true for fractional and for negative values of n, provided x is less than 1 in absolute value. The number of terms in the expansion is not finite, but is *unlimited*, and the series or expansion *converges* or approaches a definite limit as the number of terms of the expansion is increased without limit, provided $|x| < 1$.

By the above formula, we have:

$$\sqrt{1+x} = 1 + (1/2)\,x + \frac{(1/2)(1/2-1)}{2!}x^2 + \frac{(1/2)(1/2-1)(1/2-2)}{3!}x^3 + \ldots$$

$$= 1 + (1/2)x - (1/8)x^2 + (1/16)x^3 - (5/128)x^4$$

If

$$x = 1/2$$

this becomes:

$$\sqrt{3/2} = 1 + 1/4 - 1/32 + 1/128 - 5/2048 + \ldots.$$

Therefore, using five terms of the expression:

$$\sqrt{3/2} = \frac{2507}{2048} = 1.2241$$

The square root, correct to four figures, is really 1.2247. Thus the error in this case is less than one-tenth of 1 percent if only five terms of the series be used. The degree of accuracy in each case is dependent both upon the value of n and upon the value of x. Obviously, for a given value of n, the series converges for small values of x more rapidly than for larger values.

As another example, suppose it is required to expand $(1 - x)^{-1}$. By the binomial theorem:

$$(1-x)^{-1} = 1 + (-1)(-x) + \frac{-1(-1-1)}{2!}(-x)^2 + \frac{-1(-1-1)(-1-2)}{3!}(-x)^3 + \ldots$$
$$= 1 + x + x^2 + x^3 + \ldots$$

If five terms of the series be used, the error is 1/16 for $x = 1/2$, or about 3 percent.

111. Approximate Formulas. If x be very small, the expansion of:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$$

is approximately:

$$(1+x)^n \doteqdot 1 + nx \qquad (1)$$

since x^2, x^3 and all higher powers of x are much smaller than x. Thus, using the symbol $\doteqdot$ to express "approximately equals," we have, for example:

$$(1.01)^3 \doteqdot 1.03$$

for

$$(1 + 1/100)^3 \doteqdot 1 + 3/100$$

The true value of $(1.01)^3$ is 1.030301, so that the approximation is very good.

Likewise:

$$(\mathbf{1} - \mathbf{x})^n \doteqdot \mathbf{1} - \mathbf{nx} \qquad (2)$$

if x be small.

If x, y, and z be small compared with unity, the following approximate formulas hold:

$$(\mathbf{1} + \mathbf{x})(\mathbf{1} + \mathbf{y}) \doteqdot \mathbf{1} + \mathbf{x} + \mathbf{y} \qquad (3)$$

$$(\mathbf{1} + \mathbf{x})/(\mathbf{1} + \mathbf{y}) \doteqdot \mathbf{1} + \mathbf{x} - \mathbf{y} \qquad (4)$$

$$(\mathbf{1} + \mathbf{x})(\mathbf{1} + \mathbf{y})(\mathbf{1} + \mathbf{z}) \doteqdot \mathbf{1} + \mathbf{x} + \mathbf{y} + \mathbf{z} \qquad (5)$$

The approximation formulas are proved as follows:

$(1+x)(1+y) = 1 + x + y + xy \doteqdot 1 + x + y$, for xy is small compared to x and y.

$\dfrac{(1+x)}{(1+y)} = 1 + x - y + \dfrac{y^2 - xy}{1+y} \doteqdot 1 + x - y$, for the fraction is small compared to x and y.

$(1+x)(1+y)(1+z) \doteqdot (1+x+y)(1+z) \doteqdot 1 + x + y + z$

112.* The Progressive Mean. In using scientific data it is often desirable to determine the so-called **progressive mean** of a highly fluctuating magnitude. Thus if we wish to determine whether or not the rainfall at New York has on the average been increasing or decreasing in the last 100 years, we form an average for each successive group of five or six or seven or other convenient number of years, and tabulate and compare these averages. In finding these averages, however, the various years are weighted as

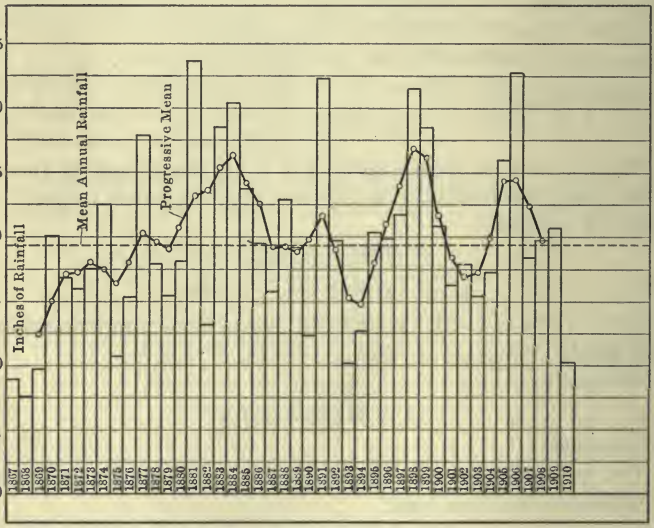


FIG. 84.—Annual, Mean, and Progressive Mean Rainfall (by 5-yr. Periods) at Dodge, Kansas.

follows: If the numbers whose progressive means are desired be $a_1, a_2, a_3, a_4, \ldots$, then the progressive mean corresponding to a_{10} would be, for five-year intervals,

$$m = (a_8 + 4a_9 + 6a_{10} + 4a_{11} + a_{12})/16$$

and for seven-year intervals,

$$m = (a_7 + 6a_8 + 15a_9 + 20a_{10} + 15a_{11} + 6a_{12} + a_{13})/64$$

In these expressions the coefficients are the binomial coefficients and the divisors are the sum of the coefficients. See Fig. 84.

Exercises

1. Explain the following approximate formulas, in which $|x| < 1$

$$\sqrt{1+x} \doteq 1 + (1/2)x$$
$$\sqrt{1-x} \doteq 1 - (1/2)x$$
$$(1+x)^{-1/2} \doteq 1 - (1/2)x$$
$$\sqrt[3]{1+x} \doteq 1 + (1/3)x$$
$$\sqrt[3]{1-x} \doteq 1 - (1/3)x$$
$$(1+x)^{-2/3} \doteq 1 - (2/3)x$$
$$(1+x^2)^{1/3} \doteq 1 + (1/3)x^2.$$

2. Compute the numerical value of:

$(1.03)^{1/2}$ $\qquad$ $(1.05)^{1/3}$

$(1.02)(1.03)$ $\qquad$ $1.02/1.03$

$(1.01)(1.02)/(1.03)(1.04)$.

3. The formula for the period of a simple pendulum is:

$$T = \pi\sqrt{l/g}$$

For the value of gravity at New York, this reduces to

$$T = \frac{\sqrt{l}}{6.253}$$

in which l, the length of the pendulum, is measured in inches. This pendulum beats seconds when

$$l = (6.253)^2 \text{ or } 39.10 \text{ inches.}$$

What is the period of the pendulum if l be lengthened to 39.13 inches?

HINT:

$$T = \frac{\sqrt{l}}{6.253}$$

$$T' = \frac{\sqrt{l+h}}{6.253} = \frac{\sqrt{l}}{6.253}\sqrt{1+h/l}$$
$$= \frac{\sqrt{l}}{6.253}(1 + h/2l)$$

Take $l = 39.10$, and $h = 0.03$

Then:

$$T' = 1 + 0.03/78.20$$
$$= 1.00038.$$

A day contains 86,400 seconds. The change of length would, therefore, cause a loss of 32.8 seconds per day, if the pendulum were attached to a clock.

4. On the ocean how far can one see at an elevation of h feet above its surface?

Call the radius of the earth $a(= 3960$ miles), and the distance one can see d, which is along a tangent from the point of observation to the sphere. Since h is in feet, and $a + \frac{h}{5280}$, d, and a are the sides of a right triangle, we have $(a + h/5280)^2 = d^2 + a^2$

or:
$$a^2(1 + h/5280a)^2 = d^2 + a^2.$$

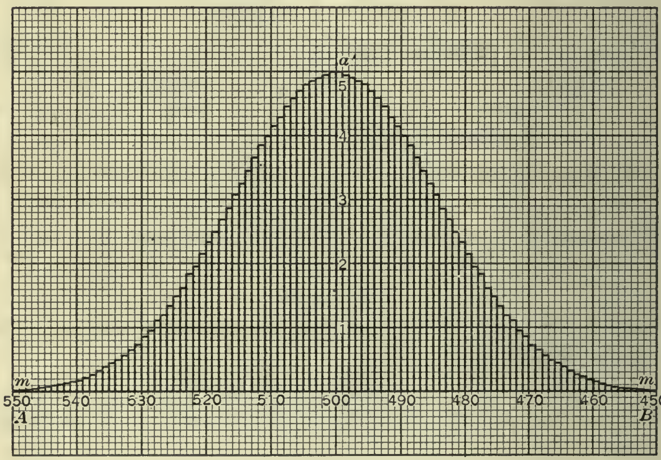


FIG. 85.—Graphical Representation of the Values of the Binomial Coefficients in the 999th power of a Binomial. The middle coefficients are taken equal to 5, for convenience, and the others are expressed to that scale also.

Expanding by the approximate formula:

$$a^2(1 + 2h/5280a) = d^2 + a^2$$

or:

$$\begin{aligned} d^2 &= 2ah/5280 \\ &= 2 \times 3960h/5280 \\ &= (3/2)h \end{aligned}$$

or:

$$d = \sqrt{(3/2)h}$$

where d is expressed in miles and h is in feet. See §**66**, exercise 13.

5. How much is the area of a circle altered if its radius of 100 cm. be changed to 101 cm.?

6. How much is the volume of a sphere, $\frac{4}{3}\pi a^3$, altered if the radius be changed from 100 cm. to 101 cm.?

7. If the formula for the horse power of a ship is I.H.P. $= \frac{S^3 D^{2/3}}{200}$ where S is speed in knots and D is displacements in tons, what increase in horse power is required in order to increase the speed from fifteen to sixteen knots, the tonnage remaining constant at 5000? What increase in horse power is required to maintain the same speed if the load or tonnage be increased from 5000 to 5500?

113.* Graphical Representation of the Coefficients of any Power of a Binomial. If we erect ordinates at equal intervals on the x-axis proportional to the coefficients of any power of a binomial, we find that a curve is approximated, which becomes very striking as the exponent is taken larger and larger. In Fig. 85, the ordinates are proportional to the coefficients of the 999th power of $(x + a)$. The drawing is due to Quetelet.

The limit of the broken line at the top of the ordinates in Fig. 85 is, as n is increased indefinitely, a bell-shaped curve, known as the **probability curve**; its equation is of the form $y = ae^{-bx^2}$, as is shown in treatises on the Theory of Probability.

CHAPTER VII

PROGRESSIONS

114. An Arithmetical Progression or an Arithmetical Series, is any succession of terms such that each term differs from that immediately preceding by a fixed number called the **common difference.** The following are arithmetical progressions:

(1) 1, 2, 3, 4, 5.
(2) 4, 6, 8, 10, 12.
(3) 32, 27, 22, 17, 12.
(4) $2\frac{1}{2}$, $3\frac{3}{4}$, 5, $6\frac{1}{4}$, $7\frac{1}{2}$.
(5) $(u - v), u, (u + v)$.
(6) $a, a + d, a + 2d, a + 3d, \ldots$

The first and last terms are called the **extremes,** and the other terms are called the **means.**

Where there are but three numbers in the series, the middle number is called the **arithmetical mean** of the other two. To find the arithmetical mean of the two numbers a and b, proceed as follows:

Let A stand for the required mean; then, by definition:

$$A - a = b - A$$

whence:

$$A = (a + b)/2$$

Thus, the arithmetical mean of 12 and 18 is 15, for 12, 15, 18 is an arithmetical progression of common difference 3.

By the **arithmetical mean** or **arithmetical average** of several numbers is meant the result of dividing the sum of the numbers by the number of the numbers. It is, therefore, such a number that if all numbers of the set were equal to the arithmetical mean, the sum of the set would be the same.

The general arithmetical progression of n terms is expressed by:

Number of term:	1	2	3	4	. . .	n
Progression:	a,	$(a + d)$,	$(a + 2d)$,	$(a + 3d)$,	. . .	$(a + [n - 1]d)$

Here a and d may be any algebraic numbers whatsoever, integral or fractional, rational or irrational, positive or negative, but n must be a positive integer. If the common difference be negative, the progression is said to be a **decreasing progression**; otherwise, an **increasing progression.**

From the general progression written above, we see that a formula for deriving the nth term of any progression may be written:

$$\mathbf{l = a + (n - 1)d} \tag{1}$$

in which l stands for the nth term.

115. The Sum of n Terms. If s stands for the sum of n terms of an arithmetical progression, and if the sum of the terms be written first in natural order, and again in reverse order, we have:

$$s = a + (a + d) + (a + 2d) + \ .\ .\ .\ + (a + [n - 1]d) \tag{1}$$

$$s = l + (l - d) + (l - 2d) + \ .\ .\ .\ + (l - [n - 1]d) \tag{2}$$

Adding (1) and (2), term by term, noting that the positive and negative common differences nullify one another, we obtain:

$$2s = (a + l) + (a + l) + (a + l) + \ .\ .\ .\ + (a + l) \tag{3}$$

or, since the number of terms in the original progression is n, we may write:

$$2s = n(a + l)$$

or:

$$\mathbf{s = n(a + l)/2} \tag{4}$$

In the above expression, $(a + l)/2$ is the average of the first and nth terms. The formula (4) states, therefore, that the sum equals the number of the terms multiplied by the average of the first and last.

116. An arithmetical progression is a very simple particular instance of a much more general class of expressions known in mathematics as **series.** A series is any sequence of terms formed according to some law, such as:

$$(x + 1) + (x + 2)^2 + (x + 3)^3, \ .\ .\ .$$

$$x + 3x^3 + 5x^5 + \ .\ .\ .$$

$$\cos x + \cos 2x + \cos 3x + \ .\ .\ .$$

It is only in a very limited number of cases that a short expression can be found for the sum of n terms of a series. An arithmetical progression is one of these exceptions.

117. The formulas (1) and (4) above are illustrated graphically by Fig. 86. Ordinates proportional to the terms of a progression are laid off at equal intervals on the line OX. The ends of these lines, because of the equal increments in the terms of the series, lie on the straight line MN. By reversing terms and adding, the sums lie within the rectangle OK whose altitude is $(a + l)$.

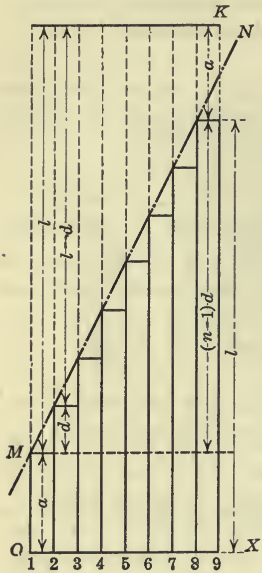


FIG. 86.—Graphical Determination of the Sum of an $A.P.$

The sum of an arithmetical progression is readily constructed. On OY, lay off the unit of measure $O1$; and, to the same scale, n. On OX, lay off $(a + l)$. From 2 on OY draw a line to $(a + l)$ on OX. From n on OY draw a parallel to the latter, cutting OX in s, the required sum. This construction has little value, except that it illustrates that s, for all values of a and d, increases indefinitely in absolute value as n increases without limit, or, using the equivalent terms already explained, that s becomes infinite as n becomes infinite.

118. Formula (1), §114, enables us to obtain the value of any one of the numbers, l, a, n, d, when three are given. Thus:

(1) Find the 100th term of:

$$3 + 8 + 13 + \ldots$$

Here: $\quad a = 3, \; d = 5, \; n = 100$

therefore, $\quad l = 3 + 99 \times 5 = 498$

(2) Find the number of terms in the progression:

$$5 + 7 + 9 + \ldots + 39$$

Here: $\quad a = 5, d = 2, l = 39$

whence: $\quad 39 = 5 + (n - 1)2$

Solving for n: $\quad n = 18$

(3) Find the common difference in a progression of fifteen terms in which the extremes are 1/2 and $42\frac{1}{2}$:

Here: $a = 1/2, l = 42\frac{1}{2}, n = 15$

whence: $42\frac{1}{2} = 1/2 + (15 - 1)d$

Solving: $d = 3$

Formula (4), **§115,** enables us to find the value of any one of the numbers s, n, a, l, when the values of the other three are given. Thus:

(5) Find the number of terms in an arithmetical progression in which the first term is 4, the last term 22, and the sum 91.

Here: $a = 4, l = 22, s = 91$

whence: $91 = n(4 + 22)/2$

solving for n: $n = 7$

The two formulas, (1) **§114** and (4) **§115,** contain five letters; hence, if any two of them stand for unknown numbers, and the values of the others are given, the values of the two unknown numbers can be found by the solution of a system of two equations. Thus:

(6) Find the number of terms in a progression whose sum is 1095, if the first term is 38 and the difference is 5.

Here: $s = 1095$, $a = 38$, and $d = 5$

whence: $l = 38 + (n - 1)5$ (1)

$1095 = n(38 + l)/2$ (2)

From (1): $l = 33 + 5n$ (3)

From (2): $2190 = 38n + nl$ (4)

Substituting the value of l from (3) in (4), we get:

$$2190 = 71n + 5n^2 \qquad (5)$$

Solving this quadratic, we find:

$$n = 15, \text{ or } - 29.2$$

The second result is inadmissible, since the number of terms cannot be either negative or fractional.

Exercises

Solve each of the following:

1. Given, $a = 7$, $d = 4$, $n = 15$; find l and s.
2. Given, $a = 17$, $l = 350$, $d = 9$; find n and s.

3. Given, $a = 3$, $n = 50$, $s = 3825$; find l and d.

4. Given, $s = 4784$, $a = 41$, $d = 2$; find l and n.

5. Given, $s = 1008$, $d = 4$, $l = 88$; find a and n.

6. Find the sum of the first n even numbers.

7. Find the sum of the first n odd numbers.

8. Insert nine arithmetical means between $-7/8$ and $+7/8$.

9. Sum $(a + b)^2 + (a^2 + b^2) + (a - b)^2$ to n terms.

10. Find the sum of the first fifty multiples of 7.

11. Find the amount of \$1.00 at simple interest at 5 percent for 1912 years.

12. How long must \$1.00 accumulate at $3\frac{1}{2}$ percent simple interest until the total amounts to \$100?

13. How many terms of the progression $9 + 13 + 17 + \ldots$ must be taken in order that the sum may equal 624? How many terms must be taken in order that the sum may exceed 750?

14. Show that the only right triangle whose sides are in arithmetical progression is the triangle of sides 3, 4, 5, or a triangle with sides proportional to these numbers.

119. Geometrical Progression. A **geometrical progression** is a series of terms such that each term is the product of the preceding term by a fixed factor called the **ratio.** The following are examples:

(1) 3, 6, 12, 24, 48.

(2) 100, -50, 25, $-12\frac{1}{2}$.

(3) 1/2, 1/4, 1/8, 1/16, 1/32.

(4) $a, ar, ar^2, ar^3, ar^4 \ldots$

The **geometrical mean** of two numbers, a and b, is found as follows: Let G stand for the required mean. Then, by the definition of a geometrical progression:

$$G/a = b/G$$

whence:

$$G^2 = ab$$

or:

$$G = \sqrt{ab}$$

Thus, 4 is the geometrical mean of 2 and 8. The *arithmetical mean* of 2 and 8 is 5. **The geometrical mean of n positive numbers** is the value of the nth root of their product. Thus the geometrical mean of:

$$8, 9 \text{ and } 24 \text{ is } 12 = \sqrt[3]{8 \times 9 \times 24}$$

120. The nth Term and the Sum of n Terms. If a represents the first term and r the ratio of any geometrical progression, the progression may be written:

Number of term:	1	2	3	4	. . .	$n-1$	n
Progression:	a,	ar,	ar^2,	ar^3,	. . .	ar^{n-2},	ar^{n-1}

Therefore, representing the nth term by l, we obtain the simple formula:

$$\mathbf{l = ar^{n-1}} \tag{1}$$

Representing by s the sum of n terms of any geometrical progression, we have:

$$s = a + ar + ar^2 + \ . \ . \ . \ + ar^{n-2} + ar^{n-1}$$

Factoring the right member:

$$s = a(1 + r + r^2 + \ . \ . \ . \ + r^{n-2} + r^{n-1})$$

But, by a fundamental theorem in factoring,[1] the expression in the parenthesis is the quotient of $1 - r^n$ by $1 - r$. Hence:

$$\mathbf{s = a(1 - r^n)/(1 - r)} \tag{2}$$

Another form is obtained by introducing l by the substitution:

$$ar^{n-1} = l$$

$$\mathbf{s = (a - rl)/(1 - r)} \tag{3}$$

121. Formula (1), or (2), enables one to find any one of the four numbers involved in the equations when three are given. The two formulas (1) and (2) considered as simultaneous equations enable one to find any two of the five numbers a, r, n, l, s, when the other three are given. But if r be one of the unknown numbers, the equations of the system may be of a high degree, and beyond the range of Chapter VII, unless solved by graphical means. If n be an unknown number, an equation of a new type is introduced, namely, one with the unknown number appearing as an exponent. Equations of this type, known as exponential equations, will be treated in the chapter on logarithms. The following examples illustrate cases in which the resulting single and simultaneous equations are readily solved.

(1) Insert three geometrical means between 31 and 496.

Here:

$$a = 31, \quad l = 496, \text{ and } n = 5$$

[1] See Appendix.

whence:

$$496 = 31 \times r^4$$

or:

$$r^4 = 16$$

therefore:

$$r = \pm 2$$

consequently, the required means are either 62, 124, and 248, or $-$ 62, $+$ 124, and $-$ 248.

(2) Find the sum of a geometrical progression of five terms, the extremes being 8 and 10,368.

Here:

$$a = 8, \quad l = 10{,}368, \text{ and } n = 5$$

whence:

$$10{,}368 = 8r^4 \qquad (1)$$

$$s = (10{,}368r - 8)/(r - 1) \qquad (2)$$

From the first,

$$r = 6$$

whence, from the second,

$$s = 12{,}440$$

(3) Find the extremes of a geometrical progression whose sum is 635, if the ratio be 2 and the number of terms be 7.

Here:

$$s = 635, \; r = 2, \text{ and } n = 7$$

whence:

$$l = a \cdot 2^6 \qquad (1)$$

$$635 = (2l - a)/1 \qquad (2)$$

Substituting l from (1) in (2), we get:

$$635 = 128\,a - a$$

whence:

$$a = 5, \text{ hence, } l = 320$$

(4) The fourth term of a geometrical progression is 4, and the sixth term is 1. What is the tenth term?

Here:

$$ar^3 = 4 \qquad (1)$$

and:

$$ar^5 = 1 \qquad (2)$$

whence, dividing (2) by (1):

$$r^2 = 1/4, \text{ or } r = \pm 1/2$$

therefore, from (1):

$$a = 4/r^3 = \pm 32$$

Then the tenth term is:

$$\pm 32(\pm 1/2)^9 = 1/16$$

Exercises

1. Find the sum of seven terms of $4 + 8 + 16 + \ldots$

2. Find the sum of $-4 + 8 - 16 + \ldots$ to six terms.

3. Find the tenth term and the sum of ten terms of $4 - 2 + 1 - \ldots$

4. Find r and s; given $a = 2$, $l = 31{,}250$, $n = 7$.

5. Insert two geometrical means between 47 and 1269.

6. Insert three geometrical means between 2 and 3.

7. Insert seven geometrical means between a^8 and b^8.

8. Show that the quotient $(a^n - b^n)/(a - b)$ is a geometrical progression.

9. Sum $x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots$ to n terms.

10. Sum $x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots$ to n terms.

11. Sum $a + ar^{-1} + ar^{-2} + \ldots$ to n terms.

12. If $a, b, c, d, \ldots$ are in geometrical progression, then $a^2 + b^2$, $b^2 + c^2$, $c^2 + d^2$, $\ldots$ are also in geometrical progression.

13. If any numbers are in geometrical progression, their differences are also in geometrical progression.

14. A man agreed to pay for the shoeing of his horse as follows: 1 cent for the first nail, 2 cents for the second nail, 4 cents for the third nail, and so on until the eight nails in each shoe were paid for. What did the last nail cost? How much did he agree to pay in all?

122. Compound Interest. Just as the amount of principle and interest of a sum of money at simple interest for n years is expressed by the $(n + 1)$st term of an arithmetical progression, so, in the same way, the amount of any sum at compound interest for n years is represented by the $(n + 1)$st term of a geometrical progression. Thus, the amount of \$1.00 at compound interest at 4 percent for twenty years is given by the expression:

$$1(1.04)^{20}$$

The amount of d dollars for n years at r percent is:

$$d\left(1 + \frac{r}{100}\right)^n$$

The present value of $1.00, due twenty years hence, estimating compound interest at 4 percent, is:

$$1/(1.04)^{20}$$

The value of $1.00, paid annually at the beginning of each year into a fund accumulating at 4 percent compound interest, is, at the end of that period:

$$(1.04)^1 + (1.04)^2 + \ . \ . \ . \ (1.04)^{20}$$

which is the sum of the terms of a geometrical progression of twenty terms.

Problems of this character in compound interest and in compound discount, and the more complicated problems that proceed therefrom, are basal to the theory of annuities, life insurance and depreciation of machinery and structures. The computation of the high powers involved necessitates the postponement of such problems until the subject of logarithms has been explained.

123. Infinite Geometrical Progressions. If the ratio of a geometrical progression be a proper fraction, the progression is said to be a **decreasing progression.** Thus:

$$1, 1/2, 1/4, 1/8, 1/16, \text{ and } 1/3, 1/9, 1/27, 1/81$$

are decreasing progressions. If we increase the number of terms in the first of these progressions the sums will always be less than 2; but the difference $2 - s$ will become and remain less than any preassigned number. By definition, 2 is, therefore, the limit of this sum.[1] The sum of n terms of this particular progression should be written down by the student for a number of successive values for n, thus:

Number of terms:

	1,	2,	3,	4,	5,	. . .	10,
Sum:	1,	$1 + 1/2$,	$1 + 3/4$,	$1 + 7/8$,	$1 + 15/16$,	. . .	$1 + 511/512$,

The nth term differs from 2 by only $1/2^{n-1}$.

It is easy to show that the sum of every decreasing geometrical progression approaches a fixed limit as the number of terms becomes infinite. For, write the formula:

$$s = \frac{a - ar^n}{1 - r}$$

[1] See definition, §80.

in the form:

$$s = \frac{a}{1 - r} - \frac{ar^n}{1 - r} \tag{1}$$

If we suppose that r is a proper fraction and that n increases without limit, then r^n can be made less than any assigned number, for the value of any power of a proper fraction decreases as the exponent of the power increases. As the other parts of the second fraction in (1) do not change in value as n changes, the fraction as a whole can be made smaller than any number that can be assigned. Hence, we write:

$$\lim_{n = \infty} s = \frac{a}{1 - r} \tag{2}$$

Exercises

As $n = \infty$, find the limit of each of the following:

1. $1/2 - 1/4 + 1/8 - 1/16 + \ . \ . \ .$

Here:

$$a = 1/2, \ r = -1/2$$

whence, the limit $s = \dfrac{1/2}{1 - (-1/2)} = 1/3.$

2. 0.3333 . . .

Here: $a = 3/10, \ r = 1/10$

whence, the limit: $s = \dfrac{3/10}{1 - 1/10} = 1/3.$

3. $9 - 6 + 4 - \ . \ . \ .$

4. 0.272727 . . .

5. 0.279279279 . . .

6. $1/3 - 1/6 + 1/12 - \ . \ . \ .$

7. $4 + 0.8 + 0.16 + \ . \ . \ .$

8. Express the number 8 as the sum of an infinite geometrical progression whose second term is 2.

124. Graphical representation of the terms and of the sum of a geometrical progression: If lines proportional to the terms of an arithmetical progression be erected at equal intervals normal to any line, the ends of the perpendiculars will lie on a straight line, as already explained in **§117**. We shall now explain a corresponding construction for a geometrical progression. First, note that all the essentials of a geometrical progression may be studied if we assume the first term to be unity, for the number a occurs only as a single constant multiplier in each term, and also occurs in the same manner in the formulas for l and s. Therefore, by taking a-fold these expressions in a geometrical series whose first term is 1, the results are obtained for the more general case.

To represent the geometrical series $1 + r + r^2 + r^3 + \ldots + r^{n-1}$ graphically, lay off $OM = 1$ on OY, $OS_1 = 1$ on OX, $S_1P_1 = r$ on the unit line, and draw MP_1. Draw the arc P_1S_2 and erect

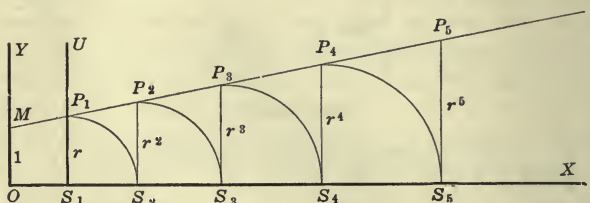


FIG. 87.—Graphical Construction of the Sum of a $G. P.$ $r > 1$.

P_2S_2. Draw the arc P_2S_2 and erect P_3S_3. Continue this construction until you draw the arc $P_{n-1}S_n$ and erect P_nS_n. The series of trapezoids OMS_1P_1, $S_1P_1P_2S_2$, $S_2P_2P_3S_3$, . . . , $S_{n-1}P_{n-1}P_nS_n$ are similar and, since $P_1S_1 = r \times OM$, it follows that $P_2S_2 = rP_1S_1$, $P_3S_3 = rP_2S_2$, . . . , $P_nS_n = rP_{n-1}S_{n-1}$. Hence we have:

$OM = OS_1 = 1$

$P_1S_1 = S_1S_2 = r \;\therefore OS_2 = 1 + r =$ sum of 2 terms

$P_2S_2 = S_2S_3 = r^2 \;\therefore OS_3 = 1 + r + r^2 =$ sum of 3 terms

$P_3S_3 = S_3S_4 = r^3 \;\therefore OS_4 = 1 + r + r^2 + r^3 =$ sum of 4 terms

---

$P_{n-1}S_{n-1} = S_{n-1}S_n = r^{n-1} \therefore OS_n = 1 + r + r^2 + \ldots r^{n-1} =$ sum of n terms.

Fig. 87 shows the series whose ratio is $r = 1.2$. Fig. 88 shows the series whose ratio is 0.8.

The line MP_1 has the slope $(r - 1)$ in Fig. 87 and the slope $-(1 - r)$ in Fig. 88. In both, its Y-intercept is 1. Its equation is, in both cases, $y = (r - 1)x + 1$ or $x = \frac{1 - y}{1 - r}$. In both figures, when $y = P_nS_n = r^n$, $x = OS_n$. Substituting these values for x and y, we get for the sum of n terms, $S = \frac{1 - r^n}{1 - r}\cdot$ Fig. 87 shows that when the number of terms is allowed to increase without limit, the sum OS_n also increases without limit. Fig. 88 shows that when the number of terms is made to increase without limit, the sum OS_n approaches OL as a limit. Now the value of OL is the value of x when $y = 0$. Hence the limit of the sum of the progression, or $OL = \frac{1}{1 - r}\cdot$

Consult also §7, problem 6, exercise 5 and Figs. 13, 14.

In Figs. 87 and 88 the ordinates OM, S_1P_1, S_2P_2, . . . representing the successive terms of the geometrical progressions, were not erected at equal intervals along OX. If the ordinates representing the successive terms of the progressions be erected at equal intervals along OX, the line $MP_1P_2P_3$ . . . passing through the ends of the ordinates will be a curve and not a straight line.

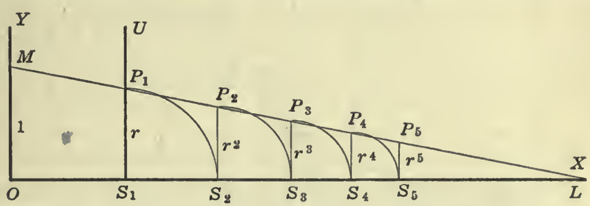


FIG. 88.—Graphical Construction of the Sum of a $G. P.$ $r < 1$.

To construct this curve, a geometrical construction different from that given above is to be preferred. Near the lower margin of a sheet of $8\frac{1}{2} \times 11$-inch unruled paper lay off a uniform scale of inches and draw vertical lines through the points of division, as shown in Fig. 89. Select one of these for the y-axis, and on the unit line lay off the given ratio of the progression $1N = r$. Then divide the y-axis proportionally to the successive powers of r, either by the method of problem 6, §7, Fig. 11, or by the method shown in Fig. 89. Through the points of division on the y-axis draw lines parallel to the x-axis, thus dividing the plane into a large number of rectangles. Starting at the point M (0, 1) sketch free hand the diagonals of successive cornering rectangles, rounding the results into a smooth curve as shown. Then the relation between ordinate y and abscissa x for the values of $x = -2, -1, 0, 1, 2, 3$, etc., is given by the equation $y = r^x$. Fig. 89 is drawn for $r = 3/2$ so that the curve is $y = (3/2)^x$.

The method used in Fig. 89 may be explained as follows: Draw the lines $y = x$ and $y = rx$. From the point $(1, r)$ on $y = rx$ draw a horizontal line to $y = x$, thence a vertical line

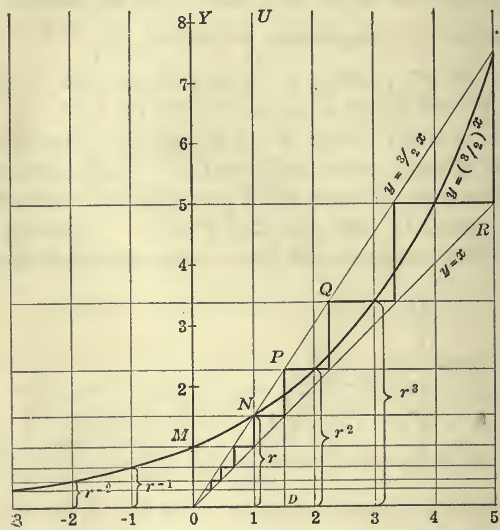


Fig. 89.—Graphical Construction of the Successive Terms of a *G. P.* In the diagram $r = 3/2$, and the curve is $y = (3/2)^x$.

to $y = rx$, etc., thereby forming the "stairway" of line segments between $y = x$ and $y = rx$ as shown in the figure. Then the points, N, P, Q, etc., have the ordinates r, r^2, r^3, etc., as required, for, to obtain the ordinate of P, or PD, the value of x used was $OD = r$, hence P is the point on $y = rx$ for $x = r$, or $y = PD = r^2$. Likewise Q is by construction the point on $y = rx$ for $x = r^2$, hence the y of the point $Q = r \times r^2 = r^3$, etc.

The figure shows the process for finding r^{-1}, r^{-2}, etc. In Chapter VIII a method will be explained for locating intermediate points on the curve.

The curve generated by the method described above is one of the most important curves in mathematics. In general, it is seen that the points located on the curve MN always satisfy an equation of the form

$$y = r^x$$

where r is a constant. This is called an **exponential equation** and the curve is known as the **exponential** or **compound interest curve.**

Note that the ordinates y to the right of M increase rapidly as x increases and that the ordinates to the left of M decrease very slowly as x decreases; that is, the curve rapidly leaves the positive x-axis, but slowly approaches the negative x-axis as an asymptote. These results are exactly reversed in case $r < 1$.

125.* Harmonical Progressions. A series of terms such that their reciprocals form an arithmetical progression are said to form an **harmonical progression.** The following are examples:

(1) $1/2, 1/3, 1/4, 1/5$.
(2) $1, 1/5, 1/9, 1/13$.
(3) $1/(x - y), 1/x, 1/(x + y)$.
(4) $1/3, 1, -1, -1/3$.
(5) $4, 6, 12$.
(6) $1/a, 1/(a + d), 1/(a + 2d), \ldots$

Although harmonical progressions are of such a simple character, no simple expression has been found for the sum of n terms. Our knowledge of arithmetical progressions enables us to find the value of any required term and to insert any required number of harmonical means between two given extremes, as in the examples below.

(1) Write six terms of the harmonical progression 6, 3, 2.

We must write six terms of the arithmetical progression, 1/6, 1/3, 1/2. The common difference of the latter is 1/6, so that the arithmetical progression is 1/6, 1/3, 1/2, 2/3, 5/6, 1, and the harmonical progression is 6, 3, 2, 1.5, 1.2, 1.

(2) Insert two harmonical means between 4 and 2.

We must insert two arithmetical means between 1/4 and 1/2; these are 1/3 and 5/12, whence the required harmonical means are 3 and 2.4.

126.* Harmonical Mean. The harmonical mean is found as follows: Let the two numbers be a and b and let H stand for the required mean. Then we have:

$$1/H - 1/a = 1/b - 1/H$$

That is:

$$2/H = 1/a + 1/b = (a + b)/ab$$

whence:

$$\mathbf{H = 2ab/(a + b)} \qquad (1)$$

Thus the harmonical mean of 4 and 12 is $96/(4 + 12) = 6$. By the **harmonical mean of several numbers** is meant the *reciprocal* of the arithmetical mean of their reciprocals. Thus the harmonical mean of 12, 8 and 48 is $13\frac{1}{11}$.

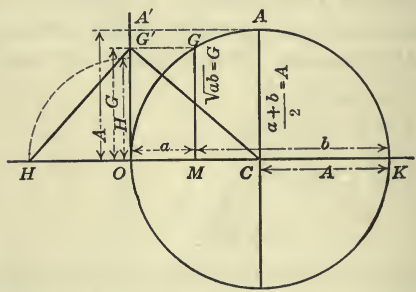


Fig. 90.—The Relation Between the Arithmetical, Geometrical and Harmonic Means.

127.* Relation between A, G, and H. As previously found:

$$A = (a + b)/2, \; G = \sqrt{ab}, \; H = 2ab/(a + b)$$

whence:

$$AH = ab$$

but:

$$ab = G^2$$

hence:

$$AH = G^2$$

or:

$$\mathbf{G = \sqrt{AH}} \qquad (1)$$

That is to say, the geometrical mean of any two positive numbers is the same as the geometrical mean of their arithmetical and harmonical means.

The arithmetical, geometrical and harmonical means may be constructed graphically as in Fig. 90. Draw the circle of diameter $(a + b) \equiv OM + MK$. Then the radius is the arithmetical mean A. Erect a perpendicular at M. Then MG is the geometrical mean. Make $OG' = MG$ and draw CG'. Draw $G'H$ perpendicular to CG'. Then OH is the harmonical mean, since

$$OG' = \sqrt{OC \times OH}$$

Now $A > G > H$; for from the figure, $MG < CA$. Therefore, the angle $G'CO$ is less than 45° and also its equal $HG'O$ is less than 45°. Therefore, $HO < OG'$ which establishes the inequality.

Exercises

1. Continue the harmonical progression 12, 6, 4.

2. Find the difference: $(1.8 + 1.2 + 0.8 + \ldots$ to 8 terms$)$
$- (1.8 + 1.2 + 0.6 + \ldots$ to 8 terms$)$.

3. If the arithmetical mean between two numbers be 1, show that the harmonical mean is the square of the geometrical mean.

CHAPTER VIII

THE LOGARITHMIC AND THE EXPONENTIAL FUNCTIONS

128. Historical Development. The almost miraculous power of modern calculation is due, in large part, to the invention of logarithms in the first quarter of the seventeenth century by a Scotchman, John Napier, Baron of Merchiston. This invention was founded on the simplest and most obvious of principles, that had been quite overlooked by mathematicians for many generations. Napier's invention may be explained as follows:[1] Let there be an arithmetical and a geometrical progression which are to be associated together, as, for example, the following:

$$\begin{array}{ccccccccccc} 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10 \\ 1, & 2, & 4, & 8, & 16, & 32, & 64, & 128, & 256, & 512, & 1024 \end{array}$$

Now the product of any two numbers of the second line may be found by adding the two numbers of the first progression above them, finding this sum in the first line, and finally taking the number lying under it; this latter number is the product sought. Thus, suppose the product of 8 by 32 is desired. Over these numbers of the second line stand the numbers 3 and 5, whose sum is 8. Under 8 is found 256, the product desired. Now since but a limited variety of numbers is offered in this table, it would be useless in the actual practice of multiplication, for the reason that the particular numbers whose product is desired would probably not be found in the second line. The overcoming of this obvious obstacle constitutes the novelty of Napier's invention. Instead of attempting to accomplish his purpose by extending the progressions by continuation at their ends, Napier proposed to insert any number of intermediate terms in each progression. Thus, instead of the portion

$$\begin{array}{ccccc} 0, & 1, & 2, & 3, & 4 \\ 1, & 2, & 4, & 8, & 16 \end{array}$$

of the two series we may write:

[1] Merely the fundamental principles of the invention, not historical details, are given in what follows.

0.	1/2,	1,	$1\frac{1}{2}$,	2,	$2\frac{1}{2}$,	3,	$3\frac{1}{2}$,	4
1,	$\sqrt{2}$,	2,	$\sqrt{8}$,	4,	$\sqrt{32}$,	8,	$\sqrt{128}$,	16

by inserting arithmetical means between the consecutive terms of the arithmetical series and by inserting geometrical means between the terms of the geometrical series. Let these be computed to any desired degree of approximation, say to two decimal places. Then we have the series

A.P.	G.P.
0.0	1.00
0.5	1.41
1.0	2.00
1.5	2.83
2.0	4.00
2.5	5.66
3.0	8.00

Again inserting arithmetical and geometrical means between the terms of the respective series we have:

A. P.	G. P.
0.00	1.00
0.25	1.19
0.50	1.41
0.75	1.69
1.00	2.00
1.25	2.38
1.50	2.83
1.75	3.36
2.00	4.00
2.25	4.76

By continuing this process each consecutive three figure number may finally be made to appear in the second column, so that, to this degree of accuracy, the product of any two such numbers may be found by the process previously explained. The decimal points of the factors may be ignored in this work, as for example, the product of 2.38×14.1 is the same as that of 238×14.1 except in the position of the decimal point. The correct position of the decimal point can be determined by inspection after the significant figures of the product have been obtained. Using the above table we find $2.38 \times 14.1 = 33.6$.

The above table, when properly extended, is a **table of logarithms.** As geometrical and arithmetical progressions different from those given above might have been used, the number of possible systems of logarithms is indefinitely great. The first column of figures contains the **logarithms** of the numbers that stand opposite them in the second column. Napier, by this process, said he divided the ratio of 1.00 to 2.00 into "100 equal ratios," by which he referred to the insertion of 100 geometrical means between 1.00 and 2.00. The "*number of the ratio*" he called the **logarithm** of the number, for example, 0.75 opposite 1.69, is the logarithm of 1.69. The word logarithm is from two Greek words meaning "*The number of the ratios.*" In order to produce a table of logarithms it was merely necessary to compute numerous geometrical means; that is, no operations except multiplication and the extraction of square roots were required. But the numerical work was carried out by Napier to so many decimal places that the computation was exceedingly difficult.

The news of the remarkable invention of logarithms induced Henry Briggs, professor at Gresham College, London, to visit Napier in 1615. It was on this visit that Briggs suggested the advantages of a system of logarithms in which the logarithm of 1 should be 0 and the logarithm of 10 should be 1, for then it would only be necessary to insert a sufficient number of geometrical means between 1 and 10 to get the logarithm of any desired number. With the encouragement of Napier, Briggs undertook the computation, and in 1617, published the logarithms of the first 1000 numbers and, in 1624, the logarithms of numbers from 1 to 20,000, and from 90,000 to 100,000 to *fourteen decimal places*. The gap between 20,000 and 90,000 was filled by a Hollander, Adrian Vlacq, whose table, published in 1628, is the source from which nearly all the tables since published have been derived.

129. Graphical Computation of Logarithms. In Fig. 89 the terms of a geometrical progression of first term 1 and ratio $1N = r$ are represented as ordinates arranged at equal intervals along OX. Fig. 89 is drawn to scale for the value of $r = 1.5$. Fig. 91 is a similar figure drawn for $r = 2$, in which a process is used for locating intermediate points of the curve, so that the locus may be sketched with greater accuracy. The lines $y = x$ and $y = rx$ (in this case $y = 2x$) are drawn as before, and the "stairway" constructed as before (see §**124**). Vertical lines drawn through $x = -2, -1, 0, 1, 2, 3, \ldots$ and horizontal lines drawn

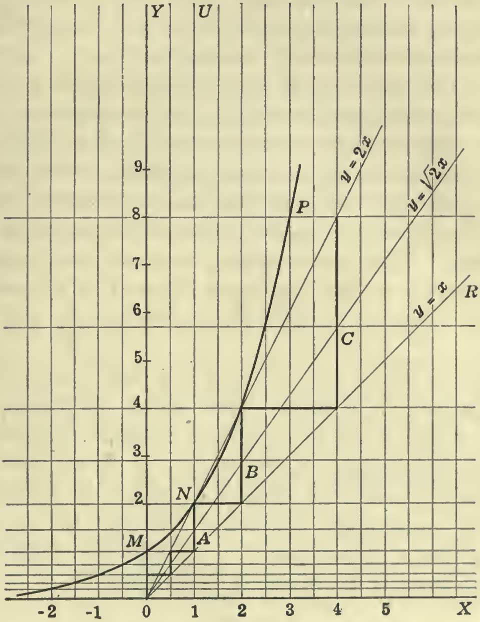


FIG. 91.—Graphical Construction of the Curve $y = 2^x$.

through the horizontal tread of each step of the stairway divides the plane into a large number of rectangles. Starting at M and sketching the diagonals of successive cornering rectangles the smooth curve MNP is drawn. Intermediate points of the curve are located by doubling the number of vertical lines by bisecting the distances between each original pair, and then by increasing the number of horizontal lines in the following manner: Draw the line $y = \sqrt{r}\,x$ (in the case of the Fig., $y = \sqrt{2}\,x$).

At the points where this line cuts the vertical risers of each step of the "stairway" (some of these points are marked A, B, C in the diagram) draw a new set of horizontal lines. Each of the original rectangles is thus divided into four smaller rectangles. Starting at M and sketching a smooth curve along the diagonals of successive cornering rectangles, the desired graph is obtained. By the use of the straight line $y = \sqrt[4]{r}\,x$ another set of intermediate points may be located, and so on, and the resulting curve thus drawn to any degree of accuracy required. In explaining this process, the student will show that the method of construction just used consists in the doubling of the number of horizontal lines of the figure by the successive insertion of geometrical means between the terms of a geometrical progression, while at the same time the number of vertical lines is successively doubled by insertion of arithmetical means between the terms of an arithmetical series. Thus the graphical work of construction of the curve corresponds to the successive insertion of geometrical and arithmetical means in the two series discussed in the preceding section.

As explained above, the ordinate y of any point of the curve MNP of Fig. 91 is a term of a geometrical progression, and the abscissa x of the same point is the corresponding term of an arithmetical progression. Since, when y is given, the value of x is determined, we say, by definition, that x is a *function* of y (§4). This particular functional relation is so important that it is given a special name: *x is called the* **logarithm** of y, and the statement is abbreviated by writing

$$x = \log y,$$

but to distinguish from the case in which some other geometrical progression might have been used, the ratio of the progression may be written as a subscript, thus:

$$x = \log_r y$$

which is read: "x is the logarithm of y to the base r."

If we assume that the process of locating the successive sets of intermediate points by the construction of successive geometrical means will lead, if continued indefinitely, to the generation of the curve MNP without breaks or gaps, then we may say that in the equation:

$$\mathbf{x} = \log_r \mathbf{y} \tag{1}$$

the logarithm is a function of y defined for all positive values of y and for all values of x.

As a matter of fact, both the arithmetical and the geometrical method given above defines the function or the curve only for rational values of x; that is, the only values of x that come into view in the process explained above are whole numbers and intermediate rational fractions like $2\frac{1}{2}$, $2\frac{1}{4}$, $2\frac{5}{8}$, $2\frac{3}{16}$, $2\frac{13}{32}$, . . .

It is seen at once from the method of construction used in Fig. 91 that the values of y at $x = 1, 2, 3, 4, \ldots$, are respectively $y = r, r^2, r^3, r^4, \ldots$, and the values of y at $x = 1/2, 3/2, 5/2, \ldots$, are $y = r^{1/2}, r^{3/2}, r^{5/2}, \ldots$, respectively, and similarly for other intermediate values of x. In other words, the equation connecting the two variables x and y may be written

$$\mathbf{y} = \mathbf{r}^x \tag{2}$$

Thus, *when the values of a variable x run over an arithmetical progression* (*of first term* 0) *while the corresponding values of a variable y run over a geometrical progression* (*of first term* 1), *the relation between the variables may be written in either of the forms* (1) *or* (2) *above.* Equation (2) is called an **exponential equation** and y is said to be an **exponential function** of x, while in (1) x is said to be a **logarithmic function** *of y.* The student has frequently been called upon in mathematics to express relations between variables in two different or "inverse" forms, analogous to the two forms $y = r^x$ and $x = \log_r y$. For example, he has written either

$$y = x^2$$

or:

$$x = \pm\sqrt{y}$$

and either

$$y = x^{n/2}$$

$$x = y^{2/n}$$

The graph of a function is of course the same whether the equation be solved for x or solved for y.

130. The student is required to construct the curves described in the following exercises by the method of **§129.** The inch, or 2 cm., may be adopted as the unit of measure; the curves should be drawn on plain paper within the interval from $x = -2$ to $x = +2$.

If tangents be drawn to the curves at $x = -2, -1, 0, 1, 2$, it will be noted, as nearly as can be determined by experiment, that the several tangents to any one curve cut the X-axis at the same constant distance to the left of the ordinate of the point of tangency. This distance is greater than unity if $r = 2$ and less than unity if $r = 3$. *The value of r for which the distance is exactly unity is later shown to be a certain irrational or incommensurable number, approximately* 2.7183 . . . , *represented in mathematics by the letter e, and called the* **Naperian base.** This number, and the number π, are two of the most important and fundamental constants of mathematics.[1]

[1] It is not easy to locate accurately the tangent to a curve at a given point of the curve. To test whether or not a tangent is correctly drawn at a point P, a number of chords parallel to the tangent may be drawn. If the two end points AB of the chord tend to approach the point of tangency P as the chord is taken nearer and nearer to P (but always parallel to AB) then the tangent was correctly drawn. If the two points A and B do not tend to coalesce at the point P when the chord is moved in the manner described, then the tangent was incorrectly drawn.

A number of instruments have been designed to assist in drawing tangents to curves. One of these, called a "Radiator," will be found listed in most catalogs

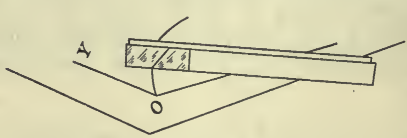


Fig. 92.—Mirrored Ruler for Drawing the Normal (and hence the Tangent) to any Curve.

of drawing instruments. Another instrument consists of a straight edge provided with a vertical mirror as shown in Fig. 92. When the straight edge is placed across a curve the reflection of the curve in the mirror and the curve itself can both be seen and usually the curve and image meet to form a cusp or angle. The straight edge may be turned, however, until the image forms a smooth continuation of the given curve. In this position the straight-edge is perpendicular to the tangent and the tangent can then be accurately drawn. See Gramberg, Technische Messungen, 1911.

Exercises

Draw the following curves on plain paper using 1 inch as the unit of measure; make the tests referred to in the second paragraph of §130.

1. Construct a curve similar to Fig. 91, representing the equation $x = \log_2 y$, from $x = -2$ to $x = +2$, and draw tangents at $x = -1$, $x = 0$, $x = 1$, $x = 2$.

2. Construct the curve whose equation is $x = \log_3 y$ from $x = -2$ to $x = +2$, and draw tangents at $x = -1$, $x = 0$, $x = 1$, $x = 2$.

3. Construct the curve whose equation is $x = \log_{2.7} y$, and show by *trial* or experiment that the tangent to the curve at $x = 2$ cuts the x-axis at nearly $x = 1$, that the tangent at $x = 1$ cuts the x-axis at nearly $x = 0$, that the tangent at $x = 0$ cuts the x-axis at nearly $x = -1$, etc.

4. Draw the curve $x = \log_{0.5} y$ and show that it is the same as the reflection of $x = \log_2 y$ in the mirror $x = 0$.

NOTE: The student must remember that the experimental testing of the properties of the tangents to the curves called for above does not constitute *mathematical proof* of the usual deductive sort familiar to him. The experimental tests have value, however, in preparing the student for the rigorous investigation of these same properties when taken up in the calculus.

131. The Exponential Function. The expression a^x, where a is any positive number except 1, has a definite meaning and value for all positive or negative rational values of x, for the meaning of numbers affected by positive or negative fractional exponents has been fully explained in elementary algebra. The process outlined above likewise defines $\log_r x$ for all rational values of x, but the process would not lead to *irrational* values of x, such as $\sqrt{2}$, $\sqrt[3]{5}$, etc. As a matter of fact the expression a^x has as yet no meaning assigned to it for irrational values of x; thus $10^{\sqrt{2}}$ has no meaning by the definitions of exponents previously given, for $\sqrt{2}$, is not a whole number, hence $10^{\sqrt{2}}$ does not mean that 10 is repeated as a factor a certain number of times; also $\sqrt{2}$ is not a fraction, so that $10^{\sqrt{2}}$ cannot mean a power of a root of 10. But if any one of the numbers of the following sequence

$$1 \qquad 1.4 \qquad 1.41 \qquad 1.414 \qquad 1.4142 \qquad 1.41421 \quad \ldots$$

be used as the exponent of 10, the resulting power can be computed to any desired number of decimal places. For example, $10^{1.41}$ is the 141th power of the 100th root of 10; to find the 100th root we may take the square root of 10, find the square root of this result, then find its 5th root, finally finding the 5th root of this last result.

If the various powers be thus computed to seven places we find:

$$
\begin{aligned}
10^{1.4} &= 25.11887\ldots\\
10^{1.41} &= 25.70396\ldots\\
10^{1.414} &= 25.94179\ldots\\
10^{1.4142} &= 25.95374\ldots\\
10^{1.41421} &= 25.95434\ldots\\
10^{1.414213} &= 25.95452\ldots\\
10^{1.4142135} &= 25.95455\ldots
\end{aligned}
$$

Now the sequence of exponents used in the first column are found by extracting the square root of 2 to successive decimal places. If the sequence in the second column approaches a limit, this limit is taken *by definition* as the value of $10^{\sqrt{2}}$. It is shown in higher mathematics that such a limit in this and similar cases always exists and consequently that a number with an irrational exponent has a meaning. In this book we shall *assume*, without a formal proof, that a^x has a meaning for irrational values of x.

To summarize: In order logically to complete the definition of a^x for irrational values of x, and to set forth other important properties, we would be required to proceed as follows:

(1) It must be shown that if x be an always rational variable approaching an irrational number n as a limit, *that the limit of a^x exists.* The notation a^x, where x is rational, is understood to mean the positive value of a^x, so that the limit of a^x, when it is shown to exist, will necessarily be a *positive number.*

(2) The above described limit of a^x must be taken as the *definition of a^n*, where n is the irrational number approached by x as a limit.

(3) It must be shown that a^x is a continuous function of x.

(4) It must be shown that the fundamental laws of exponents apply to numbers affected with irrational exponents.

When it is shown, or when it is assumed, that a value of x

always exists which will satisfy the equation $a^x = y$, where a and y are any given positive numbers, then the expression a^x is called the **exponential function of** x **with base** a; otherwise a^x is defined only for rational values of x.

132. Definitions. In the exponential equation $a^x = y$:

The number a is called the **base.**

The number y is called the **exponential function** of x to the base a, and is sometimes written $y = \exp_a x$.

The number x is called the **logarithm** of y to the base a, and is written $x = \log_a y$. Thus in the equation $a^x = y$, x may be called either **the exponent of a or the logarithm of y.**

The two equations:

$$y = a^x$$
$$x = \log_a y$$

express exactly the same relations between x and y; one equation is solved for x, the other is solved for y. The graphs are identical, just as the graphs of $y = x^2$ and $x = \pm\sqrt{y}$ are identical.

See also *Anti-logarithm*, **§142.**

133. Common Logarithms. In the equation $10^x = y$, x is called the **common logarithm of y.** It is also called the **Brigg's logarithm of y.** Thus, the common logarithm of any number is the exponent of the power to which 10 must be raised to produce the given number. Thus 2 is the common logarithm of 100, since $10^2 = 100$; likewise 1.3010 will be found to be the common logarithm of 20 correct to 4 decimal places, since $10^{1.3010} = 20.0000$ to 4 decimal places.

134. Systems of Logarithms. If in the exponential equation $y = a^x$, where a is any positive number except 1, different values be assigned to y and the corresponding values of x be computed and tabulated, the results constitute a **system of logarithms.** The number of different possible systems is unlimited, as already noted in **§128.** As a matter of fact, however, only two systems have been computed and tabulated; the **natural** or **Naperian or hyperbolic system,** whose base is an incommensurable number, approximately 2.7182818, and the **common** or **Briggs' system,** whose base is 10. The letter e is set aside in mathematics to stand for the base of the natural system.

Natural logarithms of all numbers from 1 to 20,000 have been computed to 17 decimal places. The common logarithms are usually printed in tables of 4, 5, 6, 7 or 8 decimal places.

It will be found later that the graphs of all logarithmic functions of the form $x = \log_a y$ can be made by stretching or by contracting in the same fixed ratio the ordinates of any one of the logarithmic curves. For that reason numerical tables in more than one system of logarithms are unnecessary.

In the following pages the common logarithm of any number n will be written $\log n$, and not $\log_{10} n$; that is, the base is supposed to be 10 unless otherwise designated; $\ln x$ for $\log_e x$ and $\lg x$ for $\log_{10} x$ are also used.

Exercises

Write the following in logarithmic notation.

1. $10^3 = 1000.$
2. $10^{-3} = 0.001.$
3. $10^0 = 1.$
4. $11^2 = 121.$
5. $16^{0.25} = 2.$
6. $e^x = y.$
7. $10^{0.25} = 1.7783.$
8. $10^{0.3010} = 2.$
9. $a^1 = a.$
10. $10^{\log_{10} y} = y.$

Express the following in exponential notation:

11. $\log_{10} 4 = 0.6021.$
12. $\log 10000 = 4.$
13. $\log 0.0001 = -4.$
14. $\log_2 1024 = 10.$
15. $\log_a a = 1.$
16. $\log \sqrt[3]{100} = 2/3.$
17. $\log_{27} (1/3) = -1/3.$
18. $\log_{100} 10 = 1/2.$
19. $\log 1 = 0.$
20. $\log_a 1 = 0.$

135. Graphical Table. In Fig. 93 is shown the graph of the function defined by the two progressions whose use was suggested by Briggs to Napier, and which are referred to in the last paragraph of **§128.** By inserting means three times between 0 and 1 in the arithmetical progression and between 1 and 10 in the geometrical progression, we get

A. P. or Logarithms	G. P. or Numbers	Exponential Form of G. P.
0.000	1.000	$10^{0.000}$
0.125	1.334	$10^{0.125}$
0.250	1.778	$10^{0.250}$
0.375	2.371	$10^{0.375}$
0.500	3.162	$10^{0.500}$
0.625	4.217	$10^{0.625}$
0.750	5.623	$10^{0.750}$
0.875	7.499	$10^{0.875}$
1.000	10.000	$10^{1.000}$

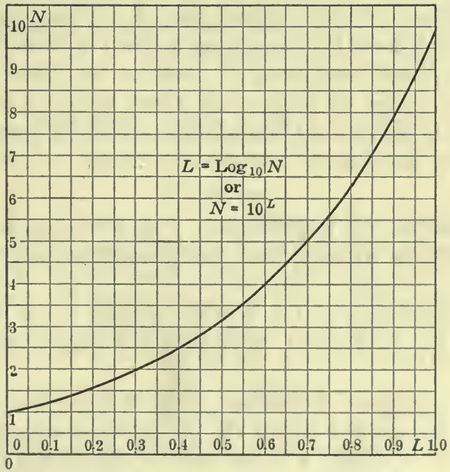


FIG. 93.—The Curve $L = \log_{10} N$.

If we let L stand for the logarithm of the number N, the functional relation is obviously $L = \log_{10} N$ or $N = 10^L$. The curve (Fig. 93) may now be used as a graphical table of logarithms from which the results can be read to about 3 decimal places.

The logarithms of numbers between 1 and 10 may be read directly from the graph. Thus, $\log_{10} 7.24 = 0.860$. If the logarithm is between 0 and 1, the number is read directly from the graph. Thus if the logarithm is 0.273, the number is 1.87.

If we multiply the readings of the N-scale by 10^n, we must add n to the readings on the L-scale, for $10^n N = 10^{L+n}$.

If we divide the readings on the N-scale by 10^n, we must subtract n from the readings on the L-scale, for $N/10^n = 10^{L-n}$.

This fact enables us to read the logarithms of all numbers from the graph, and conversely to find the number corresponding to any logarithm. Thus we have, $\log 72.4 = 1.860$, $\log 724 = 2.860$, $\log 0.724 = 0.860 - 1$, $\log 0.0724 = 0.860 - 2$.

If the logarithm is 1.273, the number is 18.7.

If the logarithm is 2.273, the number is 187.

If the logarithm is $0.273 - 1$, the number is 0.187.

If the logarithm is $0.273 - 2$, the number is 0.0187.

We observe that the computation of a three place table of logarithms would not involve a large amount of work: such a table has actually been computed in drawing the curve of Fig. 93. The original tables of Briggs and Vlacq involved an enormous expenditure of labor and extraordinary skill, or even genius in computation, because the results were given to *fourteen places of decimals.*

136. Properties of Logarithms. The following properties of logarithms follow at once from the general properties or laws of exponents.

(1) The logarithm of 1 is 0 in all systems. For $a^0 = 1$, that is, $\log_a 1 = 0$. In Fig. 91, note that the curve passes through (0, 1).

(2) The logarithm of the base itself in any system is 1. For $a^1 = 1$, that is, $\log_a a = 1$. In Fig. 91, by construction N is always the point $(1, r)$, where r is the ratio of the first or fundamental progression; in the present notation, this is the point $(1, a)$.

(3) Negative numbers have no logarithms. This follows at once from **§131,** (1). In Figs. 89, 91, and 93, note that the curves do not extend below the X-axis.

NOTE: While negative numbers have no logarithms, this does not prevent the computation of expressions containing negative factors and divisors. Thus to compute $(287) \times (-374)$, find by logarithms $(287) \times (374)$ and give proper sign to the result.

137. Logarithm of a Product. Let n and r be any two positive numbers and let:

$$\log_a n = x \text{ and } \log_a r = y \tag{1}$$

Then, by definition of a logarithm:

$$n = a^x \text{ and } r = a^y \tag{2}$$

Multiplying:

$$nr = a^x a^y = a^{x+y}$$

Therefore, by definition of a logarithm §132:

$$\log_a nr = x + y$$

or, by (1)

$$\log_a nr = \log_a n + \log_a r \tag{3}$$

Hence, *the logarithm of the product of two numbers is equal to the sum of the logarithms of those numbers.*

In the same way, if $\log_a s = z$, then:

$$nrs = a^{x+y+z}$$

that is,

$$\log_a nrs = \log_a n + \log_a r + \log_a s$$

Exercises

Find by the formulas and check the results by the curve of Fig. 93.

1. Given $\log 2 = 0.3010$, and $\log 3 = 0.4771$; find $\log 6$; find $\log 18$.

2. Given $\log 5 = 0.6990$ and $\log 7 = 0.8451$; find $\log 35$.

3. Given $\log 9 = 0.9542$, find $\log 81$.

4. Given $\log 386 = 2.5866$ and $\log 857 = 2.9330$; find the logarithm of the product.

5. Given $\log 11x = 1.888$ and $\log 11 = 1.0414$; find $\log x$.

138. Logarithm of a Quotient. Let n and r be any two positive numbers, and let:

$$\log_a n = x \quad \text{and} \quad \log_a r = y \tag{1}$$

From (1) by the definition of a logarithm,

$$n = a^x \qquad r = a^y$$

Dividing,

$$n/r = a^x \div a^y = a^{x-y}$$

Therefore by definition of a logarithm,

$$\log_a(n/r) = x - y$$

or by (1)

$$\mathbf{\log_a(n/r) = \log_a n - \log_a r} \tag{2}$$

therefore, *the logarithm of the quotient of two numbers equals the logarithm of the dividend less the logarithm of the divisor.*

Exercises

Check the results by reading them off the curve of Fig. 93.

1. Given $\log 5 = 0.6990$ and $\log 2 = 0.3010$; find $\log (5/2)$; find $\log 0.4$.

2. Given $\log 63 = 1.7993$, and $\log 9 = 0.9542$; find $\log 7$.

3. Given $\log 84 = 1.9243$ and $\log 12 = 1.0792$; find $\log 7$.

4. Given $\log 1776 = 3.2494$ and $\log 1912 = 3.2815$; find $\log 1776/1912$; find $\log 1912/1776$.

5. Given $\log x/12 = 0.4321$ and $\log 12 = 1.0792$, find $\log x$.

139. Logarithm of any Power. Let n be any positive number and let:

$$\log_a n = x \tag{1}$$

From (1), by the definition of a logarithm,

$$n = a^x$$

Raising both sides to the pth power, *where p is any number whatsoever*,

$$n^p = a^{px}$$

therefore, by definition of a logarithm,

$$\log_a(n^p) = px$$

or by (1):

$$\mathbf{\log_a(n^p) = p \log_a n} \tag{2}$$

therefore *the logarithm of any power of a number equals the logarithm of the number multiplied by the index of the power.*

The above includes as special cases, (1) the finding of the logarithm of any integral power of a number, since in this case p is a positive integer, or (2) the finding of the logarithm of any root of a number, since in this case p is the reciprocal of the index of the root.

Exercises

1. Given $\log 2 = 0.3010$; find $\log 1024$; find $\log \sqrt{2}$; find $\log \sqrt[3]{2}$.

2. Given $\log 1234 = 3.0913$; find $\log \sqrt{1234}$. Find $\log \sqrt[5]{1234}$.

3. Given $\log 5 = 0.6990$; find $\log 5^{2/3}$; find $\log 5^{3/2}$.

4. Simplify the expression $\log 30/\sqrt{210}$.

Express by the principles established in §§**137–139** the following logarithms in as simple a form as possible:

5. $\log (\sqrt[3]{9} \div \sqrt{3})$.

6. $\log (\sqrt[3]{12} \div \sqrt{6})$.

7. $\log (u^{1/2} \div u^{2/3})$.

8. $\log (10a^2b^3/a^{1/2}\, b^2)$.

9. Show that $\log (11/15) + \log (490/297) - 2 \log (7/9) = \log 2$.

10. Find an expression for the value of x from the equation $3^x = 567$.

SOLUTION: Take the logarithm of each side

$$x \log 3 = \log 567$$

But $\log 567 = \log (3^4 \times 7) = 4 \log 3 + \log 7$

therefore:

$$x \log 3 = 4 \log 3 + \log 7$$

or:

$$x = 4 + (\log 7)/(\log 3).$$

11. Find an expression for x in the equation $5^x = 375$.

12. Given $\log 2 = 0.3010$ and $\log 3 = 0.4771$, find how many digits in 6^{10}.

13. Find an expression for x from the equation:

$$3^x \times 2^{x+1} = \sqrt{512}.$$

14. Prove that $\log (75/16) - 2 \log (5/9) + \log (32/243) = \log 2$.

140. Characteristic and Mantissa. The common logarithm of a number is always written so that it consists of a positive decimal part and an integral part which may be either positive or negative. Thus $\log 0.02 = \log 2 - \log 100 = 0.3010 - 2$. Log 0.02 is *never written* -1.6990.

When a logarithm of a number is thus arranged, special names are given to each part. The positive or negative integral part is called the **characteristic** of the logarithm. The positive decimal part is called the **mantissa.** Thus, in $\log 200 = 2.3010$, 2 is the characteristic and 3010 is the mantissa. In $\log 0.02 = 0.3010 - 2$, (-2) is the characteristic and 3010 is the mantissa.

Since $\log 1 = 0$ and $\log 10 = 1$, every number lying between 1 and 10 has for its common logarithm a proper fraction—that is, the characteristic is 0. Thus $\log 2 = 0.3010$, $\log 9.99 = 0.9996$, $\log 1.91 = 0.281$. Starting with the equation:

$$\log 1.91 = 0.2810$$

we have, by **§137**,

$$\begin{aligned} \log 19.1 &= \log 1.91 + \log 10 &&= 0.2810 + 1 \\ \log 191 &= \log 1.91 + \log 100 &&= 0.2810 + 2 \\ \log 1910 &= \log 1.91 + \log 1000 &&= 0.2810 + 3, \text{ etc.} \end{aligned}$$

Likewise, by **§138**,

$$\begin{aligned} \log\ 0.191 &= \log 1.91 - \log 10 &&= 0.2810 - 1 \\ \log\ 0.0191 &= \log 1.91 - \log 100 &&= 0.2810 - 2 \\ \log\ 0.00191 &= \log 1.91 - \log 1000 &&= 0.2810 - 3, \text{ etc.} \end{aligned}$$

Since the characteristic of the common logarithm of any number having its first significant figure in units place is zero, and since moving the decimal point to the right or left is equivalent to multiplying or dividing by a power of 10, or equivalent to adding an integer to or subtracting an integer from the logarithm, (**§135**): (1) the value of the characteristic is dependent merely upon the position of the decimal point in the number; (2) the value of the mantissa is the same for the logarithms of all numbers that differ only in the position of the decimal point. In particular, we derive therefrom the following rule for finding the characteristic of the common logarithm of any number:

The characteristic of the common logarithm of a number equals the number of places the first significant figure of the number is removed from units' place, and is positive if the first significant figure stands to the left of units' place and is negative if it stands to the right of units' place.

Thus in $\log 1910 = 3.2810$, the first figure 1 is three places from units' place and the characteristic is 3. In $\log 0.0191 = 0.2810 - 2$ the first significant figure 1 is two places to the right of units' place and the characteristic is -2. A computer in determining the characteristic of the logarithm of a number first points to units place and counts *zero*, then passes to the next place and counts *one* and so on until the first significant figure is reached.

Logarithms with negative characteristics, like $0.3010 - 1$, 0.3010 − 2, etc., are frequently written in the equivalent form 9.3010 − 10, 8.3010 − 10, etc.

Exercises

1. What numbers have 0 for the characteristic of their logarithm? What numbers have 0 for the mantissa of their logarithms?

2. Find the characteristics of the logarithms of the following numbers: 1234, 5,678,910, 212, 57.45, 345.543, 7, 7.7, 0.7, 0.00000097, 0.00010097.

3. Given that log 31,416 = 4.4971, find the logarithms of the following numbers: 314.16, 3.1416, 3,141,600, 0.031416, 0.31416, 0.00031416.

4. Given that log 746 = 2.8727, write the numbers which have the following logarithms: 4.8727, 1.8727, 0.8727 − 3, 0.8727 − 1, 3.8727, 0.8727 − 4.

141. Logarithmic Tables. A table of logarithms usually contains only the *mantissas* of the logarithms of a certain convenient sequence of numbers. For example, a four place table will contain the mantissas of the logarithms of numbers from 100 to 1000; a five place table will usually contain the mantissas of the logarithms of numbers from 1000 to 10,000, and so on. Of course it is unnecessary to print decimal points or characteristics.

A table of logarithms should contain means for readily obtaining the logarithms of numbers intermediate to those tabulated, by means of **tabular differences** and **proportional parts.**

The tabular differences are the differences between successive mantissas. If any tabular difference be multiplied successively by the numbers 0.1, 0.2, 0.3, . . . , 0.8, 0.9, the results are called the **proportional parts.** Thus, from a four place table we find log 263 = 2.4200. The tabular difference is given in the table as 16. If we wish the logarithm of 263.7, the proportional part 0.7×16 or 11.2 is added to the mantissa, giving, to four places, log 263.7 = 2.4211. This process is known as **interpolation.** Corrections of this kind are made with great rapidity after a little practice. It is obvious that the principle used in the correction is the equivalent of a geometrical assumption that the graph of the function is *nearly straight* between the successive values of the argument given in the table. *The corrections* *should invariably be added mentally and all the work of interpolation should be done mentally if the finding of the proportional parts by mental work does not require multiplication beyond the range of* 12×12. *To make interpolations mentally* is an essential practice, if the student is to learn to compute by logarithms with any skill beyond the most rudimentary requirements.

A good method to follow is as follows: Suppose log 13.78 is required. First write down the characteristic 1; then, with the table at your left, find 137 in the number column and mark the corresponding mantissa by placing your thumb above it or your first finger below it. *Do not read this mantissa*, but read the tabular difference, 32. From the p. p. table find the correction, 26, for 8. Now return to the mantissa marked by your finger, and read it increased by 26, *i.e.*, 1393; then place 1393 after the characteristic 1 previously written down.

The accuracy required for nearly all engineering computations does not exceed 3 or 4 significant figures. Four figure accuracy means that the errors permitted do not exceed 1 *percent of* 1 *percent*. Only a small portion of the fundamental data of science is reliable to this degree of accuracy.[1] The usual measurements of the testing laboratory fall far short of it. Only in certain work in geodesy, and in a few other special fields of engineering, should more than four place logarithms be used.

142. Anti-logarithms. If we wish to find the number which has a given logarithm, it is convenient to have a table in which the logarithm is printed *before* the number. Such a table is known as a table of **anti-logarithms.** It is usually not best to print tables of anti-logarithms to more than four places; to find a number when a five place logarithm is given, it is preferable to use the table of logarithms inversely, as the large number of pages required for a table of anti-logarithms is a disadvantage that is not compensated for by the additional convenience of such a table.

[1] Fundamental constants upon which much of the calculation in applied science must be based are not often known to four figures. The mechanical equivalent of heat is hardly known to 1 percent. The specific heat of superheated steam is even less accurately known. The tensile, tortional and compressive strength of no structural material would be assumed to be known to a greater accuracy than the above-named constants. Of course no calculated result can be more accurate than the least accurate of the measurements upon which it depends.

143. Cologarithms. Any computation involving multiplication, division, evolution and involution may be performed by the addition of a single column of logarithms. This possibility is secured by using the **cologarithm,** instead of the logarithm, of all divisors. The cologarithm, or complementary logarithm, of a number n is defined to be $(10 - \log n) - 10$. The part $(10 - \log n)$ can be taken from the table just as readily as $\log n$, *by subtracting in order all the figures of the logarithm, including the characteristic,* from 9, *except the last figure, which must be taken from* 10. The subtraction should, of course, be done mentally. Thus $\log 263 = 2.4200$, whence $\text{colog } 263 = 7.5800 - 10$. It is obvious that the *addition* of $(10 - \log n) - 10$ is the same as the *subtraction* of $\log n$.

The convenience arising from this use may be illustrated as follows:

Suppose it is required to find x from the proportion

$$37.4^2 : x :: 647 : \sqrt{0.582}.$$

We then have

$$\begin{aligned} 2 \log 37.4 &= 3.1458 \\ (1/2) \log 0.582 &= 9.8825 - 10 \\ \text{colog } 647 &= \underline{7.1891} - 10 \\ \log [1.650] &= 0.2174 \end{aligned}$$

Therefore $x = 1.650$.

It is a good custom to enclose a *computed* result in square brackets.

144. Arrangement of Work. All logarithmic work should be arranged in a vertical column and should be done with pen and ink. Study the formula in which numerical values are to be substituted and decide upon an arrangement of your work in the vertical column which will make the additions, subtractions, etc., of logarithms as systematic and easy as possible. Fill out the vertical column with the names and values of the data before turning to the table of logarithms. This is called **blocking out the work.** The work is not properly blocked out unless every entry in the work as laid out is carefully labelled, stating exactly the name and value of the magnitude whose logarithm is taken, and unless the computation sheet bears a formula or statement fully explaining the purpose of the work.

Computation Sheet, Form $M7$, is suitable for general logarithmic computation.

Exercises

1. From a four place table find the logarithms of the following numbers: 342, 1322, 8000, 872.4, 35.21, 0.00213, 3.301, 325.67, $2\frac{3}{4}$, 3.1416, 0.0186, 250.75, 0.0007, 0.33333.

2. Find the numbers corresponding to each of the following logarithms: 0.3250, 2.1860, 0.8724, 1.1325, 3.0075, $8.3990 - 10$, $9.7481 - 10$, 4.0831, $7.0091 - 10$, 0.5642.

3. Compute by logarithms the value of the following: $2.56 \times 3.11 \times 421$; $7.04 \times 0.21 \times 0.0646$; $3215 \times 12.82 \div 864$.

4. Compute the following by logarithms: $81^3 \div 17^4$; $158\sqrt[4]{0.52}$; $(343/892)^3$; $\sqrt{1893}\ \sqrt{1912/446^2}$.

5. Compute the following by logarithms: $(2.7182)^{1.408}$; $(7.41)^{-5/}$; $(8.31)^{0.27}$.

6. Solve the following equations: $5^x = 10$; $3^{x-1} = 4$; $\log_x 71 = 1.21$; $\log_x 5 = \log_{10} 4.822$.

7. Find the amount of \$550 in fifteen years at 5 percent compound interest.

8. A corporation is to repay a loan of \$200,000 by twenty equal annual payments. How much will have to be paid each year, if money be supposed to be worth 5 percent?

Let x be the amount paid each year. As the debt of \$200,000 is owed *now*, the *present value* of the twenty equal payments of x dollars each must add up to the debt or \$200,000. The sum of x dollars to be paid n years hence has a present worth of only

$$\frac{x}{(1.05)^n}$$

if money be worth 5 percent compound interest. The present value, then, of x dollars paid one year hence, x dollars paid two years hence, and so on, is

$$\frac{x}{1.05} + \frac{x}{(1.05)^2} + \frac{x}{(1.05)^3} + \cdots + \frac{x}{(1.05)^{20}}$$

This is a geometrical progression.

The result in this case is the value of an annuity payable at the end of each year for twenty years that a present payment of \$200,000 will purchase.

9. It is estimated that a certain power plant costing \$220,000 will become entirely worthless except for a scrap value of \$20,000 at the end of twenty years. What annual sum must be set aside to amount to the cost of replacement at the end of twenty years, if 5 percent compound interest is realized on the money in the depreciation fund?

Let the annual amount set aside be x. In this case the twenty equal payments are to have a value of \$200,000 *twenty years hence,* while in the preceding problem the payments were to be worth \$200,000 *now.* In this case, therefore,

$$x(1.05)^{19} + x(1.05)^{18} + x(1.05)^{17} + \ldots + x(1.05)^2 + x(1.05) + x = \$200{,}000.$$

The geometrical progression is to be summed and the resulting equation solved for x.

10. The population of the United States in 1790 was 3,930,000 and in 1910 it was 93,400,000. What was the average rate percent increase for each decade of this period, assuming that the population increased in geometrical progression with a uniform ratio for the entire period.

11. Find the surface and the volume of a sphere whose radius is 7.211.

12. Find the weight of a cone of altitude 9.64 inches, the radius of the base being 5.35 inches, if the cone is made of steel of specific gravity 7.93.

13. Find the weight of a sphere of cast iron 14.2 inches in diameter, if the specific gravity of the iron be 7.30.

14. In twenty-four hours of continuous pumping, a pump discharges 450 gallons per minute; by how much will it raise the level of water in a reservoir having a surface of 1 acre? (1 acre = 43560 sq. ft.)

145. Trigonometric Computations. Logarithms of the trigonometric functions are used for computing the numerical value of expressions containing trigonometric functions, and in the solution of triangles. The right triangles previously solved by use of the natural functions are often more readily solved by means of logarithms. (See §66.) The tables of logarithmic functions contain adequate explanation of their use, so that detailed instructions need not be given in this place. Two new matters of great importance are met with in the use of the logarithms of the trigonometric functions that do not arise in the use of a table of logarithms of numbers, which, on that account, require especial attention from the student:

(1) In interpolating in a table of logarithms of trigonometric functions, *the corrections to the logarithms of all co-functions must be subtracted and not added.* Failure to do this is the cause of most of the errors made by the beginner.

(2) To secure proper relative accuracy in computation, the S and T functions must be used in interpolating for the sine and tangent of small angles.

In the following work, four place tables of logarithms are supposed to be in the hands of the students.

Exercises

1. A right prism, whose base is a square 17.45 feet on a side, is cut by a plane making an angle of 27° 15′ with a face of the prism. Find the area of the section of the prism made by the cutting plane.

2. The perimeter of a regular decagon is 24 feet. Find the area of the decagon.

3. To find the distance between two points B and C on opposite banks of a river, a distance CA is measured 300 feet, perpendicular to CB. At A the angle CAB is found to be 47° 27′. Find the distance CB.

4. In running a line 18 miles in a direction north, 2° 13.2′ east, how far in feet does one depart from a north and south line passing through the place of beginning?

5. How far is Madison, Wisconsin, latitude 43° 5′, from the earth's axis of rotation, assuming that the earth is a sphere of radius 3960 miles?

6. Find the length of the belt required to connect an 8-foot and a 3-foot pulley, their axes being 21 feet apart.

7. A man walking east 7° 15′ north along a river notices that after passing opposite a tree across the river he walks 107 paces before he is in line with the shadow of the tree. Time of day, noon. How far is it across the river?

8. Solve the right-angled triangle in which one leg $= 2\sqrt{3}$ and the hypotenuse $= 2\pi$.

9. The moon's radius is 1081 miles. When nearest the earth, the moon's apparent diameter (the angle subtended by the moon's disk as seen from the position of the earth's center) is 32.79′. When farthest from the earth, her apparent diameter is only 28.73′. Find the nearest and farthest distances of the moon in miles.

10. A pendulum 39 inches long vibrates 3° 5′ each side of its mean position. At the end of each swing, how far is the pendulum bob above its lowest position?

11. If the deviation of the compass be 2° 1.14′ east, how many feet does magnetic north depart from true north in a distance of 1 mile true north?

12. Solve:

$$x : 1.72 = 427 : \sqrt{2gh}$$

if $g = 32.2$ and $h = 78.2$.

13. The four strings of a violin are tuned in fifths; that is, for two vibrations of any string there are three vibrations of the next higher string. If the lowest or G string vibrates 196 times per second, find the number of vibrations per second of the highest string.

14. A substance containing 20 percent of impurities is to be purified by crystallization from a mother liquid. Each crystallization reduces the impurity 88.6 percent. How many crystallizations will produce a substance 0.9999 pure?

15. Compute the value of $(1 - ae^{-by})^n$ where $a = 15.6$, $b = \frac{2\pi}{\lambda}$, $\lambda = 10$, $n = 2$, $y = 2.5$.

16. Find the volume of a cone if the angle at the apex be 15° 38′ and the altitude 17.48 inches.

17. The angle subtended by the sun's diameter as seen from the earth is 32′.06. Find the diameter of the sun in miles, if the distance from the earth to the sun be 92.8 million miles.

18. Compute by logarithms four values of p from the equation $p = 32.2d^{1.408}$, for $d = 2, 3, 4, 5$.

19. Solve $3^x = 405$ for the value of x.

20. Compute:

$$\frac{23.07 \times 0.1354 \times \sqrt{234}}{13.54}.$$

What advantage is there in using the co-logarithm of the denominator?

146. Logarithmic and Exponential Curves. The graphical construction of the exponential curve has already been explained. It was noted that curves whose equations are of the form $y = r^x$ pass through the point (0, 1) and that the slope of the curves for positive values of x is steeper the larger the value selected for the number r. See Fig. 94. In a system of exponential curves $y = r^x$ passing through the point (0, 1) or the point M of Fig. 94, we shall assume that there is one curve passing through M with slope 1. The equation of this particular curve we shall call $y = e^x$, thereby *defining the number e as that value of r for which the curve* $y = r^x$ *passes through the point* (0, 1) *with slope* 1. This is a *second* definition of the number e; we shall show in this section that it is consistent with the first definition of e given in **§130.**

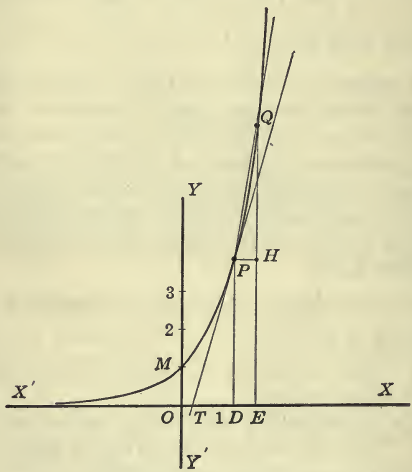


FIG. 94.—Definition of *Tangent* to a Curve.

The exercises of **§130** developed experimentally the characteristic property of the exponential curve to the base e: *The slope of the curve* $y = e^x$ *at any point is equal to the ordinate of that point.* This fact, developed experimentally in **§130**, will now be shown to follow necessarily from the definition of e just given.

Select the point P on the curve $y = e^x$ at any point desired.

Draw a line through P cutting the curve at any neighboring point Q. (Fig. 94.) A line like PQ that cuts a curve at two points is called a *secant* line. As the point Q is taken nearer and nearer to the point P (P remaining fixed), the limiting position approached by the secant PQ is called the **tangent** to the curve at the point P. This is the general definition of the tangent to any curve.

The slope of the secant joining P to the neighboring point Q is HQ/PH. As the point Q approaches P this ratio approaches the slope of the tangent to $y = e^x$ at the point P. Let $OD = x$ and $PH = h$; then $OE = x + h$, also $DP = e^x$ and $EQ = e^{x+h}$. Since HQ is the y of the point Q minus the y of the point P, we have:

$$\frac{HQ}{PH} = \frac{e^{x+h} - e^x}{h} = e^x \frac{e^h - 1}{h}$$

Now the slope of $y = e^x$ at P is the limit of the above expression as Q approaches P or as h approaches zero. That is:

$$\text{slope of } e^x \text{ at } P = e^x \lim_{h \doteq 0} \frac{e^h - 1}{h} \tag{1}$$

We now seek to find

$$\lim_{h \doteq 0} \frac{e^h - 1}{h}$$

if such limit exists. Since P is any point, consider the point M where $x = 0$. The slope there is:

$$e^0 \lim_{h \doteq 0} \frac{e^h - 1}{h}$$

That is, the slope of $y = e^x$ at M is:

$$\lim_{h \doteq 0} \frac{e^h - 1}{h}$$

But by the definition of e, the slope of $y = e^x$ at M is 1. Hence we must conclude that the required limit exists and that

$$\lim_{h \doteq 0} \frac{e^h - 1}{h} = 1 \tag{2}$$

Substituting this result in equation (1), we have

$$\text{Slope at } P = e^x \tag{3}$$

This expresses the fact that the slope of $y = e^x$ at any point is e^x, or is the ordinate y of that point, a fact that was first indicated experimentally in §**130**. At that same place the approximate value of e was seen to be 2.7. A more exact value is known to be 2.7183, as will be computed later.

In Fig. 94 the slope of $y = \epsilon^x$ at P is given by PD measured by the unit OM. The distance TD, called the **subtangent,** is constant for all positions of the point P.

The slope of $y = r^x$ at any point is readily found. There exists a number m such that $e^m = r$. Hence $y = r^x$ may be written $y = (e^m)^x = e^{mx}$. Now this curve is made from $y = e^x$ by substituting mx for x, or by multiplying all of the abscissas of the latter by $1/m$. Therefore the side TD of the triangle PDT in Fig. 94 will be multiplied by $1/m$, the other side DP remaining

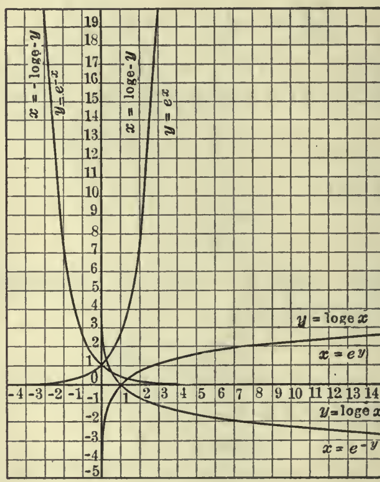


FIG. 95.—Exponential and Logarithmic Curves to the Natural Base $e = 2.7183$.

the same. Therefore the slope of the curve, or DP/TD will be multiplied by m, since the denominator of this fraction is multiplied by $1/m$. Hence the slope of $y = r^x$ at any point is m times the ordinate of that point, where m satisfies the equation $e^m = r$.

The curve $y = e^{-x}$ is, of course, the curve $y = e^x$ reflected in the Y-axis.[1] This curve, as well as the curve $y = \log_e x$ and its symmetrical curve, are shown in Fig. 95. Sometimes the curve $y = e^x$ is called the **exponential curve** and the curve $y = \log_e x$ is called the **logarithmic curve.** This distinction, however, has

[1] See **§24.**

little utility, as the equation of either locus can be expressed in either notation.

The notation $\mathbf{y} = \mathbf{ln\ x}$ is often used to indicate the *natural logarithm of x* and the notation $\mathbf{y} = \mathbf{lg\ x}$ or $\mathbf{y} = \mathbf{log\ x}$ is used to stand for the *common logarithm of x*.

TABLE IV.

The following table of powers of e is useful in sketching exponential curves.

$e^{0.2} = 1.2214$	$e^{-0.2} = 0.8187$
$e^{0.4} = 1.4918$	$e^{-0.4} = 0.6703$
$e^{0.6} = 1.8221$	$e^{-0.6} = 0.5488$
$e^{0.8} = 2.2255$	$e^{-0.8} = 0.4493$
$e = 2.7183$	$e^{-1} = 0.3679$
$e^{2} = 7.3891$	$e^{-2} = 0.1353$
$e^{3} = 20.0855$	$e^{-3} = 0.0498$
$e^{4} = 54.5982$	$e^{-4} = 0.0183$

$e^{1/2} = 1.6487$
$e^{1/3} = 1.3956$
$e^{1/4} = 1.2840$
$e^{1/5} = 1.2214$

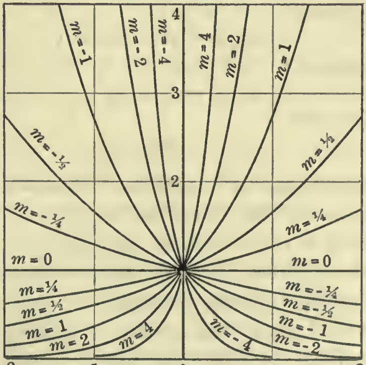


FIG. 96.—A Family of Exponentials, $y = l^{mx}$.

Exercises

1. Draw the curve $y = e^x + e^{-x}$. Show that y is an *even* function of x, that is, that y does not change when the sign of x is changed.

2. Draw the curve $y = e^x - e^{-x}$. Show that this is an odd function of x, that is, that the function changes sign but not absolute value when the sign of x is changed.

3. Draw the graphs of $y = e^{x/2}$, and $y = e^{x/2}$.

4. Draw the graphs of $y = e^{x/2}$, and $y = e^{-x/2}$.

5. Compare the curves: $y = e^{x/4}$, $y = e^{x/2}$, $y = e^x$, $y = e^{3x}$.

6. Sketch the curves $y = 1^x$, $y = 2^x$, $y = 3^x$, $y = 4^x$, $y = 5^x$, $y = 6^x$, $y = 8^x$, $y = 10^x$, from $x = -3$ to $x = +3$.

7. From the graphs of

$$y = x^2$$

and

$$y = \log_{10} x + 1.8$$

solve the equation

$$x^2 - \log x - 1.8 = 0.$$

8. Solve graphically the equation

$$5 \log x - (1/2)x + 2 = 0.$$

9. Solve graphically:

$$10^x = x^2.$$

10. Solve graphically:

$$(1/2)^x = \log x.$$

11. Solve graphically:

$$10^x = 5 \cdot \sin x.$$

12. Solve graphically:

$$\sin x = x - 0.1.$$

13. Solve graphically:

$$\cos x = x^2 - 1.$$

14. Solve analytically:

$$e^{x-1} = 10^x.$$

147. The Exponential Curve and the Theorems on Loci. It has already been shown (§ **145**) that the curve $y = a^x$ can be derived from the curve $y = e^x$ $(a > e)$ by multiplying the abscissas of the latter curve by $1/m$ $(m > 1)$, that is, by orthographic projection of $y = e^x$ upon a plane passing through the Y-axis. There exists a number m $(m > 1)$ such that $a = e^m$. Hence, $y = a^x$ may be written $y = e^{mx}$ and, by §**27**, the latter curve may be made from $y = e^x$ by multiplying its abscissas by $1/m$. Also note that the slope of the curve $y = e^x$ at any point is equal to the ordinate of the point, and that the slope of $y = a^x$ at any point is m times the ordinate of that point. *The number* $1/m$ *is called the* **modulus** *of the logarithmic system whose base is* a.

The modulus of the common system is the reciprocal of the value of m that satisfies $e^m = 10$, or it is the value of M that satisfies $e^{1/M} = 10$, or that satisfies $e = 10^M$. That is, the modulus M of the common system is the logarithm of e to the base 10, or, to four figures, equals 0.4343. The value of m or $1/M = 2.3026$. Thus we have the fundamental formulas:

$$\left.\begin{aligned} 10^{0.4343} &= e \\ e^{2.3026} &= 10 \end{aligned}\right\} \tag{1}$$

and

$$\left.\begin{aligned} \log_{10} N &= 0.4343 \log_e N \\ \log_e N &= 2.3026 \log_{10} N \end{aligned}\right\} \tag{2}$$

Another remarkable property of the logarithmic curve appears from comparing the curves $y = a^x$ and $y = a^{x+1}$, or, more generally, the curves $y = a^x$ and $y = a^{x+h}$. The second of these curves can be derived from $y = a^x$ by translating the latter curve the distance 1 (in the general case the distance h) to the left. But $y = a^{x+1}$ may be written $y = aa^x$, and $y = a^{x+h}$ may be written $y = a^h a^x$. From these it can be seen that the new curves may also be considered as derived from $y = a^x$ by multiplying all ordinates of $y = a^x$ by a, or in the general case, by a^h.

Translating the exponential curve in the x-direction is the same as multiplying all ordinates by a certain fixed number, or is equivalent to a certain orthographic projection of the original curve upon a plane through the X-axis.

Changing the sign of h changes the sense of the translation and changes elongation to shortening or *vice versa*.

The exponential curve might be defined as the locus that possesses the above-described fundamental property. There are numerous ways in which this property may be stated. Another form is this: Any portion of the exponential curve included within any interval of x, may be made from the portion of the curve included within any other equal interval of x, by the elongation (or shortening) of the ordinates in a certain ratio, or, in other words, by orthographic projection upon a plane passing through the x-axis. This is illustrated by Fig. 97, which is a graph of an exponential curve drawn to base 2. If the portions of the curve P_1P_2, P_2P_3, P_3P_4, . . . corresponding to equal intervals 1 of x

be changed by shortening all ordinates of P_1P_2 measured above the height of P_1 in the ratio 1/2, by shortening all ordinates of P_2P_3 measured above P_2 in the ratio 1/4, by shortening all ordinates of P_3P_4 measured above P_3 in the ratio 1/8, . . . the results are the curves P_1F_1, P_2F_2, P_3F_3, . . . which are identical with the portion P_0P_1 of the original curve.

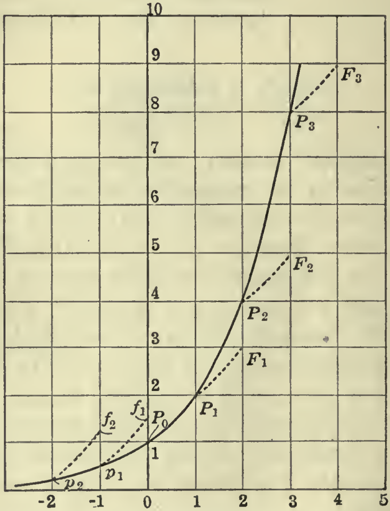


FIG. 97.—Illustration of an Important Property of the Exponential Curve.

This is also illustrated by Fig. 93, which is a small portion of the curve $x = \log_{10} y$ drawn on a large scale, and, for convenience, with the vertical unit 1/10 the horizontal unit. From this small portion of the curve we may read the logarithms of all numbers. For the distances along the x-axis may be designated 0.0, 0.1, 0.2, . . . or 1.0, 1.1, 1.2, . . . or 2.0, 2.1, 2.2, . . ., etc., in which case we read 1, 2, 3, . . . or 10, 20, 30, . . . or 100, 200, 300, . . . etc., respectively, along the y-axis. This, it will be observed, is merely a geometrical statement of the fact that a table of mantissas for the numbers from 1.000 to 9.999 is sufficient for determining the logarithms of all four-figure numbers.

Exercises

1. State the difference between the curves $y = e^x$ and $y = 10^x$.

2. Graph $y = e^{-0.1t}$ where $e = 2.7183$.

3. Graph the logarithmic spiral $\rho = e^{\theta}$, θ being measured in radians.

NOTE: The radian measure in the margin of Form $M3$ should be used for this purpose.

4. Graph $\rho = e^{-\theta}$.

5. The pressure of the atmosphere is given in millimeters of mercury by the formula:

$$y = 760 \cdot e^{-x/8000}$$

where the altitude x is measured in meters above the sea level. Produce a table of pressure for the altitudes $x = 0$; 10; 50; 100; 200; 300; 1000; 10,000; 100,000.

6. From the data of the last problem, find the pressure at an altitude of 25,000 feet.

7. Show that the relation of Exercise 5 may be written:

$$x = 18{,}421\,(\log 760 - \log y).$$

8. Determine the value of the quotient $\dfrac{\ln x}{\lg x}$ for the following values of x: 2, 3, 5, 7.

9. How large is $e^{0.001}$ approximately?

10. What is the approximate value of $10^{0.001}$?

148. Logarithmic Double Scale. The relation between a number and its logarithm can be shown by a double scale of the sort discussed in §§**3** and **8**. In constructing the double scale, one may select for the uniform scale either the one on which the numbers are to be read, or the one on which the logarithms are to be read. A scale having a most remarkable and useful property results if the logarithms are laid off on a uniform scale and the corresponding numbers are laid off on a non-uniform scale, as shown in the double scale of Fig. 98. This scale is constructed for the base 10. The distances measured on the B-scale, although it is the scale on which the numbers are read, *are proportional to the common logarithms of the successive numbers*; that is, if the total length of the scale be called unity, the distance on the B scale from the left end to the mark 2 is 0.3010, the distance to the mark 3 is 0.4771, etc.; also the distance on this scale from the left end to the mark 6 is the sum of the distance from the left end to the mark 2 and the like distance to the mark 3; also the distance to 8 is just treble the distance to 2.

FIG. 98.—Logarithmic Double Scale.

Since $\log 10x = 1 + \log x$, it follows that, if the scales A and B, Fig. 98, were extended another unit to the right, this second unit would be identical to the first one, except in the attached numbers. The numbers on the A-scale would be changed from 0.0, 0.1, 0.2, . . . 1.0 to 1.0, 1.1, 1.2, . . ., 2.0, while those on the non-uniform, or B-scale, would be changed from 1, 2, 3, . . ., 10 to 10, 20, 30, . . . 100.

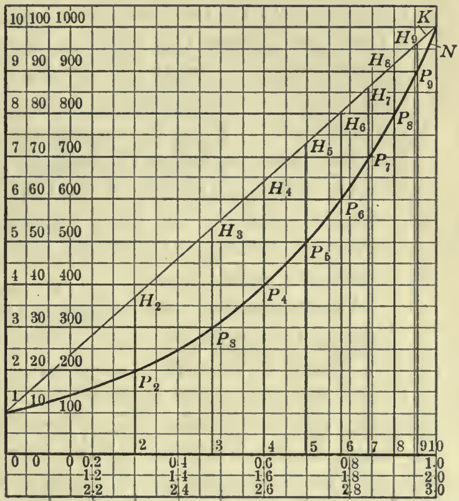
Passing along this scale an integral number of unit intervals corresponds thus to change of characteristic in the logarithms, or to change of decimal point in the numbers.

It is not, however, necessary to construct more than one block of this double scale, since we are at liberty to add an integer n to the numbers of the uniform scale, provided at the same time we multiply the numbers of the non-uniform scale by 10^n. In this way we may obtain any desired portion of the extended scale. Thus, we may change 0.1, 0.2, 0.3, . . ., 1.0 on A to 3.1, 3.2, 3.3, . . ., 4.0, by adding 3 to each number, provided at the same time we change the numbers on the B-scale 1, 2, 3, 4, . . ., 10 to 1000, 2000, 3000, 4000, . . . 10,000 by multiplying them by 10^3. If n is negative (say -2) we may write, as in the case of logarithms, $8.0 - 10$, $8.1 - 10$, $8.2 - 10$, . . ., $9.0 - 10$, or, more simply, -2, -1.9, -1.8, -1.7, . . ., -1.0, changing the numbers on the non-uniform scale at the same time to 0.01, 0.02, 0.03, . . ., 0.10.

To produce the scale of distances proportional to the logarithms of the successive numbers as used above, it is merely necessary to draw horizontal lines through the points 1, 2, 3, . . . of the y-axis in Fig. 99, and then draw vertical lines through the points

P_2, P_3, P_4 . . . where the horizontal lines meet the curve; the intercepts on the x-axis are then proportional to $\log x$.

149. The Slide Rule. By far the most important application of the non-uniform scale ruled proportionally to $\log x$, is the computing device known as the **slide rule.** The principle upon which the operation of the slide rule is based is very simple. If we have



From 1 to 2 is 3 spaces then 2 – 4 4 – 8

FIG. 99.—A Method of Constructing the Logarithmic Scale.

two scales divided proportionally to $\log x$ (A and B, Fig. 100), so arranged that one scale may slide along the other, then by sliding one scale (called the slide) until its left end is opposite any desired division of the first scale, and, selecting any desired division of the slide, as at R, Fig. 100, taking the reading of the original scale beneath this point, as N, the product of the two factors whose logarithms are proportional to AB and BR can be read directly from the lower scale at N; for AN is, by construction, the sum of AB and BR, and since the scales were laid off proportionally to $\log x$, and marked with the numbers of which the distances are the logarithms, the process described adds the logarithms mechanically, but indicates the results in terms of the numbers themselves. By this device all of the operations commonly carried out by use of a logarithmic table may be performed mechanically.

Fig. 100.—The Theory of the Slide Rule.

Full description of the use of the slide rule need not be given in detail at this place, as complete instructions are found in the pamphlets furnished with each slide rule. A very brief amount of individual instruction given to the student by the instructor will insure the rapid acquirement of skill in the use of the instrument. In what follows, the four scales of the slide rule are designated from top to bottom of the rule, A, B, C, D, respectively. The ends of the scales are called the **indices.**

An ordinary 10-inch slide rule should give results accurate to three significant figures, which is accurate enough for most of the purposes of applied science.

An exaggerated idea sometimes prevails concerning the degree of accuracy required by work in science or in applied science. Many of the fundamental constants of science, upon which a large number of other results depend, are known only to three decimal places. In such cases greater than three figure accuracy is impossible even if desired. In other cases greater accuracy is of no value even if possible. The real desideratum in computed results is, first, *to know by a suitable check that the work of computation is correct,* and, second, *to know to what order or degree of accuracy both the data and the result are dependable.*

The absurdity of an undue number of decimal places in computation is illustrated by the original tables of logarithms, which if now used would enable one to compute from the radius of the earth, the circumference *correct to* 1/10,000 *part of an inch.*

The following matters should be emphasized in the use of the slide rule:

(1) All numbers for the purpose of computation should be considered as given with the first figure in units place. Thus 517 $\times$ 1910 $\times$ 0.024 should be considered as 5.17 $\times$ 1.19 $\times$ 2.4 $\times$ $10^2 \times 10^3 \times 10^{-2}$. The result should then be mentally approximated (say 24,000) for the purpose of locating the decimal point, and for checking the work.

(2) A proportion should always be solved by one setting of the slide.

(3) A combined product and quotient like

$$\frac{a \times b \times c \times d}{r \times s \times t}$$

should always be solved as follows:

Place runner on a of scale D.
Set r of scale C to a of scale D;
Runner to b of C;
s of C to runner;
Runner to c of C;
t of C to runner;
at d of C find on D the significant figures of the result.

(4) The runner must be set on the first half of A for square roots of odd numbered numbers, and on the second half of A for the square roots of even numbered numbers.

(5) Use judgment so as to compute results in most accurate manner—thus instead of computing 264/233, compute 31/233 and hence find $264/233 = 1 + 31/233$.[1]

(6) Besides checking by mental calculation as suggested in (1) above, also check by computing several neighboring values and graphing the results if necessary. Thus check 5.17 $\times$ 1.91 $\times$ 2.4 by computing both 5.20 $\times$ 19.2 $\times$ 2.42 and 5.10 $\times$ 1.90 $\times$ 2.38.

Exercises

Compute the following on the slide rule.

1. 3.12 $\times$ 2.24; 1.89 $\times$ 4.25; 2.88 $\times$ 3.16; 3.1 $\times$ 236.

2. 8.72/2.36; 4.58/2.36; 6.23/2.12; 10/3.14.

3. 32.5 $\times$ 72.5; 0.000116 $\times$ 0.00135; 0.0392/0.00114.

4. 3,967,000 $\div$ 367,800,000.

5. $\dfrac{6.54 \times 42.6}{32.5}$; $\dfrac{8.75 \times 5.25}{32.3}$.

[1] Show by trial that this gives a more accurate result.

6. $\dfrac{78.5 \times 36.6 \times 20.8}{5.75 \times 29.5}$.

7. $\dfrac{6.46 \times 57.5 \times 8.55}{3.26 \times 296 \times 0.642}$.

8. Solve the proportion

$$x : 1.72 : = : 4.14 : \sqrt{2gh}$$

where $g = 32.2$ and $h = 78.2$.

9. Compute $\dfrac{\sqrt{171} \times 1.41}{166.7 \times 4.5}$.

10. The following is an approximate formula for the area of a segment of a circle:

$$A = h^3/2c + 2ch/3$$

where c is the length of the chord and h is the altitude of the segment.

Test this formula for segments of a circle of unit radius, whose arcs are $\pi/3$, $\pi/2$, and π radians, respectively.

11. Two steamers start at the same time from the same port; the first sails at 12 miles an hour due south, and the second sails at 16 miles an hour due east. Find the bearing of the first steamer as seen from the second (1) after one hour, (2) after two hours, and compute their distances apart at each time.

The following exercises require the use of the data printed herewith. An "acre-foot" means the quantity of water that would cover 1 acre 1 foot deep. "Second-foot" means a discharge at the rate of 1 cubic foot of water per second. By the "run-off" of any drainage area is meant the quantity of water flowing therefrom in its surface stream or river, during a year or other interval of time.

1 square mile = 640 acres
1 acre = 43,560 square feet.
1 day = 86,400 seconds.
1 second foot = 2 acre feet per day.
1 cubic foot = $7\frac{1}{2}$ gallons.
1 cubic foot water = $62\frac{1}{2}$ pounds water.
1 h.p. = 550 foot pounds per second.
450 gallons per minute = 1 second foot.

Each of the following problems should be handled on the slide rule as a continuous piece of computation.

12. A drainage area of 710 square miles has an annual run-off of 120,000 acre feet. The average annual rainfall is 27 inches. Find what percent of the rainfall appears as run-off.

13. A centrifugal pump discharges 750 gallons per minute against a total lift of 28 feet. Find the theoretical horse power required. Also daily discharge in acre feet if the pump operates fourteen hours per day.

14. What is the theoretical horse power represented by a stream discharging 550 second feet if there be a fall of 42 feet?

15. A district containing 25,000 acres of irrigable land is to be supplied with water by means of a canal. The average annual quantity of water required is $3\frac{1}{2}$ feet on each acre. Find the capacity of the canal in second feet, if the quantity of water required is to be delivered uniformly during an irrigation season of five months.

16. A municipal supply amounts to 35,000,000 gallons per twenty-four hours. Find the equivalent in cubic feet per second.

17. A single rainfall of 3.9 inches on a catchment area of 210 square miles is found to contribute 17,500 acre feet of water to a storage reservoir. The run-off is what percent of the rainfall in this case?

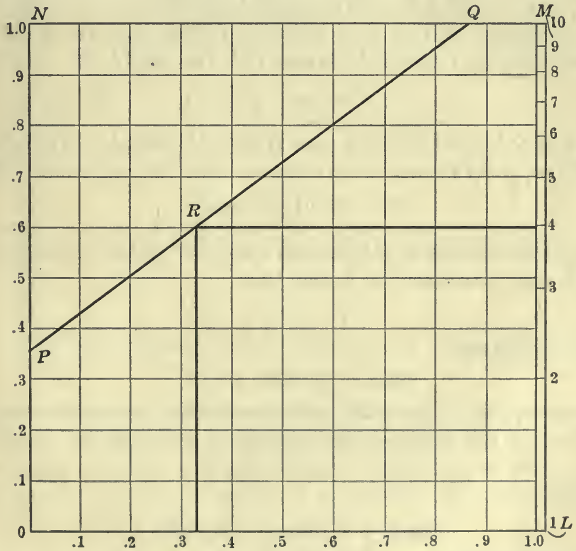


FIG. 101.—The Theory of the Use of Semi-logarithmic Paper.

150. Semi-logarithmic Coördinate Paper. Fig. 101 represents a sheet of rectangular coördinate paper, on which ON has been chosen as the unit of measure. Along the right-hand edge of this sheet is constructed a logarithmic scale LM of the type discussed in § **148**, *i.e.*, any number, say 4, on the scale LM stands opposite the logarithm of that number (in the case named opposite 0.6021) on the uniform scale ON.

Let us agree always to designate by capital letters distances measured on the uniform scales, and by lower case letters distances measured on the logarithmic scale. Thus Y will mean the ordinate of a point as read on the scale ON, while y will mean the ordinate of a point as read on the scale LM. In other words, we agree to plot a function, using logarithms of the values of the function as ordinates and the natural values of the argument or variable as abscissas.

Let PQ be any straight line on this paper, and let it be required to find its equation, referred to the uniform x-scale OL and the logarithmic y-scale LM. We proceed as follows:

The equation of this line, referred to the uniform X-axis OL and the uniform Y-axis ON, where O is the origin, is

$$Y = mX + B$$

m being the slope of the line, and B its y-intercept. Now, for the line PQ, $m = 0.742$ and $B = 0.36$, so that the equation of PQ is

$$Y = 0.742X + 0.36 \tag{1}$$

To find the equation of this curve referred to the scales LM and OL, it is only necessary to notice that

$$Y = \log y$$

so that we obtain:

$$\log y = 0.742x + 0.36 \tag{2}$$

The intercept 0.36 was read on the scale ON, and is therefore the logarithm of the number corresponding to it on the scale LM. That is, $0.36 = \log 2.30$. Substituting this value in equation (2) we obtain:

$$\log y = 0.742x + \log 2.30$$

which may be written

$$\log y - \log 2.30 = 0.742x$$

or,

$$\log \frac{y}{2.30} = 0.742x$$

On changing to exponential notation this becomes:

$$\frac{y}{2.30} = 10^{0.742x}$$

or,

$$y = 2.30(10^{0.742x}) \tag{3}$$

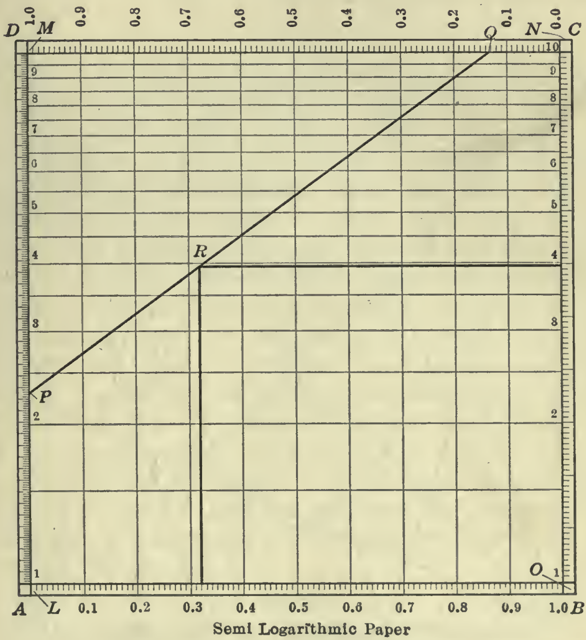


FIG. 102.—Illustration of Squared Paper, Form $M5$. The finer rulings of Form $M5$ have been omitted in Fig. 102.

In general, if the equation of a straight line referred to the scales OL and ON is

$$Y = mX + B \tag{4}$$

its equation referred to the scales OL and LM may be obtained by replacing Y by $\log y$ and B by $\log b$ in the manner described above, giving

$$\log y = mx + \log b \tag{5}$$

which, as above, may be reduced to the form

$$y = b10^{mx} \tag{6}$$

This is the general equation of the exponential curve. Hence: *Any exponential curve can be represented by a straight line, provided ordinates are read from a suitable logarithmic scale, and abscissas are read from a uniform scale.*

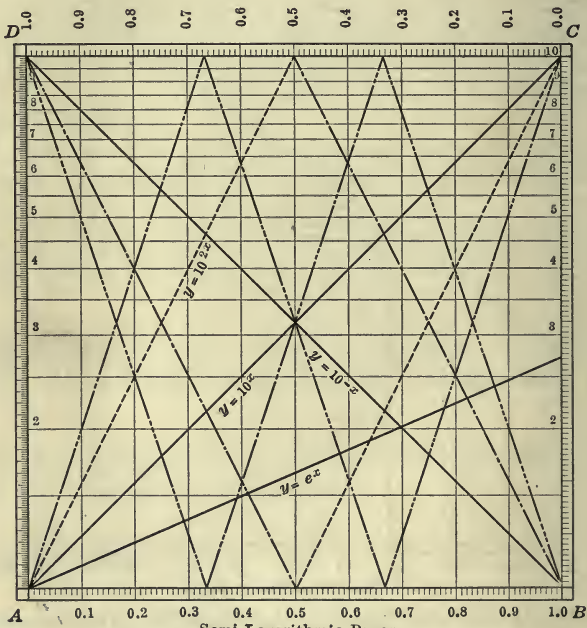


FIG. 103.—Exponential Curves on Form M5. The curve — . . . is $y = 10^{-3x}$; — . is $y = 10^{-2x}$; — . . is $y = 10^{3x}$.

Fig. 102 represents the same line PQ ($y = (2.30)10^{0.742x}$), as Fig. 101. The two figures differ only in one respect: in Fig. 101 the rulings of the uniform scale ON are extended across the page, while in Fig. 102 these rulings are replaced by those of the scale LM.

Coördinate paper such as that represented by Fig. 102 is known as **semi-logarithmic paper.** It affords a convenient coördinate system for work with the exponential function.

Every point on PQ (Fig. 102) satisfies the exponential equation

$$y = 2.30(10^{0.742x})$$

Thus, in the case of the point R,

$$3.98 = 2.30(10^{0.742})^{0.320}$$
$$= 2.30(10^{0.238})$$

The slope of any line on the semi-logarithmic paper may be read or determined by means of the uniform scales BC and AB of form $M5$. The scale AD of form $M5$ is the scale of the natural logarithms, so that any equation of the form $y = e^{mx}$ can be graphed at once by the use of this scale. Thus, the line $y = e^x$ (Fig. 103) passes through the point A or (0, 1), and a point on BC opposite the point marked 1.0 on AD. Note that 1.0 on scale AD 2.718 on the non-uniform scale of the main body of the paper and 0.4343 on the scale BC all fall together, as they should.

To draw the line $y = 10^{-x}$, the corner D of the plate may be taken as the point (0, 1). On the line drawn once across the sheet representing $y = 10^{mx}$, y has a range between 1 and 10 only. To represent the range of y between 10 and 100, two or more sheets of form $M5$ may be pasted together, or, preferably, the continuation of the line may be shown on the same sheet by suitably changing the numbers attached to the scales AB and BC. Thus Fig. 103 shows in this manner $y = 10^{2x}$ and $y = 10^{3x}$.

Remember that the line

$$\mathbf{y} = \mathbf{b}10^{mx} \tag{7}$$

passes through the point (0, b) with slope m. Note that

$$\frac{\mathbf{y}}{\mathbf{b}} = 10^{m(x - a)} \tag{8}$$

passes through the point (a, b) with slope m.

Exercises

On semi-logarithmic paper draw the following:

1. $y = 10^{3x}$, $y = 10^{2x}$, $y = 10^x$, $y = 10^{-x}$, $y = 10^{-2x}$, $y = 10^{-3x}$.

2. $y = e^{2x}$, $y = e^x$, $y = e^{-x}$, $y = e^{-2x}$.

3. $3x = \log y$, $(1/2)x = \log y$.

4. $y = 10^{x/2}$, $y = 10^{x/10}$.

5. Graph $y = 2(10)^x$ and $\frac{y}{2} = 10^{x-3}$.

151. The Compound Interest Law. Logarithmic Increment. The law expressed by the exponential curve was called by Lord Kelvin the **compound interest law** and since that time this name has been generally used. It is recalled that the exponential curve was drawn by using ordinates equal to the successive terms of a geometrical progression which are uniformly spaced along the x-axis; since the amount of any sum at compound interest is given by a term of a geometrical progression, it is obvious that a sum at compound interest accumulates by the same law of growth as is indicated by a set of uniformly spaced ordinates of an exponential curve; hence the term "compound interest law," from this superficial view, is appropriate. The detailed discussion that follows will make this clear:

Let \$1 be loaned at r percent per annum compound interest. At the end of one year the amount is: $(1 + r/100)$.
At the end of two years the amount is: $(1 + r/100)^2$
and at the end of t years it is: $(1 + r/100)^t$.
If interest be compounded semi-annually, instead of annually, the amount at the end of t years is: $(1 + r/200)^{2t}$
and if compounded monthly the amount at the end of the same period is: $(1 + r/1200)^{12t}$
or if compounded n times per year $y = (1 + r/100n)^{nt}$
where t is expressed in years. Now if we find the *limit* of this expression as n is increased indefinitely, we will find the amount of principle and interest on the hypothesis that the interest was *compounded continuously*. For convenience let $r/100n = 1/u$. Then:

$$y = (1 + 1/u)^{urt/100}. \tag{1}$$

where the limit is to be taken as u or n becomes infinite. Calling

$$(1 + 1/u)^u = f(u) \tag{2}$$

and expanding by the binomial theorem for any integral value of u, we obtain:

$$f(u) = 1 + u(1/u) + \frac{u(u-1)}{1.2}\,\frac{1}{u^2} + \ldots$$

$$= 1 + 1 + (1 - 1/u)/2! + (1 - 1/u)(1 - 2/u)/3! + \ldots \tag{3}$$

In the calculus it is shown that the limit of this series as u becomes infinite is the limit of the series

$$1 + 1 + 1/2! + 1/3! + \ . \ . \ . \qquad (4)$$

The limit of this series is easily found; it is, in fact, the Napierian base e. It is shown in the calculus that the restriction that u shall be an integer may be removed, so that the limit of (3) may be found when u is a continuous variable.

It is easy to see that the limit of (4) is $> 2\frac{1}{2}$ and < 3. The sum of the first three terms of the series (4) equals $2\frac{1}{2}$; the rest of the terms are positive, therefore $e > 2\frac{1}{2}$. The terms of the series (4), after the first three, are also observed to be less, term for term, than the terms of the progression:

$$(1/2)^2 + (1/2)^3 + \ . \ . \ . \qquad (5)$$

But this is a geometrical progression the limit of whose sum is 1/2. Therefore (3) is always less than $2\frac{1}{2} + \frac{1}{2}$ or 3. The value of e is readily approximated by the following computation of the first 8 terms of (4):

$$\begin{array}{rl}
 & 2.00000 = 1 + 1 \\
3| & \underline{0.50000} = 1/2! \\
4| & \underline{0.16667} = 1/3! \\
5| & \underline{0.04167} = 1/4! \\
6| & \underline{0.00833} = 1/5! \\
7| & \underline{0.00139} = 1/6! \\
 & \underline{0.00020} = 1/7! \\
\text{Sum of 8 terms} = & 2.71826
\end{array}$$

The value of e here found is correct to four decimal places.

Returning to equation (1) above, the amount of \$1 at r percent compound interest *compounded continuously is:*

$$y = e^{rt/100}$$

Thus \$100 at 6 percent compound interest, compounded annually, amounts, at the end of ten years, to

$$y = 100(1.06)^{10} = \$179.10$$

The amount of \$100 *compounded continuously* for ten years is

$$y = 100e^{0.6} = \$182.20$$

The difference is thus \$3.10.

The compound interest law is one of the important laws of nature. As previously noted, the slope or rate of increase of the exponential function

$$y = ae^{bx}$$

at any point is always proportional to the ordinate or to the value of the function at that point. *Thus when in nature we find any function or magnitude that increases at a rate proportional to itself* we have a case of the expontential or compound interest law. The law is also frequently expressed by saying, as has been repeatedly stated in this book, that *the first of two magnitudes varies in geometrical progression while a second magnitude varies in arithmetical progression.* A familiar example of this is the increased friction as a rope is coiled around a post. A few turns of the hawsers about the bitts at the wharf is sufficient to hold a large ship, because as the number of turns increases in arithmetical progression, the friction increases in geometrical progression. Thus the following table gives the results of experiments to determine what weight could be held up by a one-pound weight, when a cord attached to the first weight passed over a round peg the number of times shown in the first column of the table:

n = number of turns of the cord on the peg	w = weight just held in equilibrium by one-pound weight	Logs of preceding numbers	d = logarithmic increment
1/2	1.6	0.204	....
1	3.0	0.477	0.273
$1\frac{1}{2}$	5.1	0.709	0.231
2	8.0	0.903	0.195
$2\frac{1}{2}$	14.0	1.146	0.243
3	23.0	1.362	0.216

Average logarithmic increment = 0.23

If the weights sustained were exactly in geometrical progression, their logarithms would be in arithmetical progression. The test for this fact is to note whether the differences between logarithms of successive values are constant. These differences are known as **logarithmic increments** or in case they are negative, as **logarithmic decrements.** In the table the logarithmic increments fluctuate about the mean value 0.23.

The equation connecting n and w is of the form

$$w = 10^{n/m} \text{ or } n = m \log w$$

By graphing columns 1 and 3 on squared paper, the value of m is determined and we find

$$w = 10^{0.45n} \text{ or } n = 2.2 \log 2$$

Another way is to graph columns 1 and 2 on semi-logarithmic paper.

An interesting example of the compound interest law is Weber's law in psychology, which states that if stimuli are in geometrical progression, the sense perceptions are in arithmetical progression.

152. Modulus of Decay, Logarithmic Decrement. In a very large number of cases in nature the "compound interest" law appears as a decreasing function rather than as an increasing function, so that the equation is of the form

$$y = ae^{-bx} \tag{1}$$

where $-b$ is essentially negative. The following are examples of this law:

(1) If the thickness of panes of glass increase in arithmetical progression, the amount of light transmitted decreases in geometrical progression. That is, the relation is of the form

$$L = ae^{-bt} \tag{2}$$

where t is the thickness of the glass or other absorbing material and L is the intensity of the light transmitted. Since when $t = 0$ the light transmitted must have its initial intensity, L_0, (2) becomes

$$L = L_0e^{-bt} \tag{3}$$

The constant b must be determined from the data of the problem. Thus, if a pane of glass absorbs 2 percent of the incident light,

$$L_0 = 100, L = 98 \text{ for } t = 1,$$

then: $$98 = 100e^{-b}$$

or $$\log 98 - \log 100 = -b \log e.$$

Therefore: $$b = \frac{0.0088}{0.4343} = 0.02$$

The light transmitted by ten panes of glass is then

$$L_{10} = 100e^{-10(0.02)} = 100e^{-0.2}$$

or, by the table of §**146**,

$$L_{10} = 100/1.2214 = 82 \text{ percent}$$

(2) Variation in atmospheric pressure with the altitude is usually expressed by Halley's Law:

$$p = 760e^{-h/8000}$$

where h is the altitude in meters above sea level and p is the atmospheric pressure in millimeters of mercury. See §**147**, Exercises 5, 6, 7.

(3) Newton's law of cooling states that a body surrounded by a medium of constant temperature loses heat at a rate proportional to the difference in temperature between it and the surrounding medium. This, then, is a case of the compound interest law. If θ denotes temperature of the cooling body above that of the surrounding medium at any time t, we must have

$$\theta = ae^{-bt}$$

The constant a must be the value of θ when $t = 0$, or the initial temperature of the body.

Exercises

1. A thermometer bulb initially at temperature 19°.3 C. is exposed to the air and its temperature θ observed to be 14°.2 C. at the end of twenty seconds. If the law of cooling be given by $\theta = \theta_0 e^{-bt}$ where t is the time in seconds, find the value of θ and b.

Solution: The condition of the problem gives $\theta = 19.3$ when $t = 0$, hence $\theta_0 = 19.3$. Also, $14.2 = 19.3e^{-20b}$. This gives

$$\log 19.3 - 20b \log e = \log 14.2$$

from which b can be readily computed.

2. If $1\frac{1}{2}$ percent of the incident light is lost when light is directed through a plate of glass 0.3 cm. thick, how much light would be lost in penetrating a plate of glass 2 cm. thick?

3. Forty percent of the incident light is lost when passed through a plate of glass 2 inches thick. Find the value of a in the equation $L = L_0 e^{-at}$ where t is thickness of the plate in inches, L is the percent of light transmitted, and $L_0 = 100$.

4. As I descend a mountain the pressure of the air increases each foot by the amount due to the weight of the layer of air 1 foot thick. As the density of this layer is itself proportional to the pressure, show that the pressure as I descend must increase by the compound interest law.

5. Power is transmitted in a clock through a train of gear wheels n in number. If the loss of power in each pair of gears is 10 percent, draw a curve showing the loss of power at the nth gear.

NOTE: The graphical method of §**124**, Figs. 88, 89, may appropriately be used.

6. Given that the intensity of light is diminished 2 percent by passing through one pane of glass, find the intensity I of the light after passing through n panes.

7. The number of bacteria per cubic centimeter of culture increases under proper conditions at a rate proportional to the number present. Find an expression for the number present at the end of time t if there are 1000 per cubic centimeter present at time zero, and 8000 per cubic centimeter present at time 10.

8. The temperature of a body cooling according to Newton's law fell from 30° to 18° in six minutes. Find the equation connecting the temperature of the body and the time of cooling.

153. Empirical Curves on Semi-logarithmic Coördinate Paper. One of the most important uses of semi-logarithmic paper is in determining the functional relation between observed data, when such data are connected by a relation of the exponential form. Suppose, for example, that the following are the results of an experiment to determine the law connecting two variables x and y:

x	0.04	0.18	0.36	0.51	0.685	0.833	0.97
y	5.3	4.4	3.75	3.1	2.6	2.33	1.9

If the equation connecting x and y is of the exponential form, the points whose coördinates are given by corresponding values of x and y in the table will lie in a straight line, except for such slight errors as may be due to inaccuracies in the observations. Plotting the points on semi-logarithmic coördinate paper, we find that they lie nearly on the line PQ (Fig. 104). Assuming that, if the data were exact, the points would lie exactly on this line,[1] we may pro-

[1] We would not be at liberty to make such an assumption if the variation of the points away from the line was of a character similar to that represented by the dots near the top of Fig. 104. These points, although not departing greatly from the line shown, depart from it systematically. That is, they lie below it at each end, and above it in the center, seeming to approximate a curve, (such as the one shown dotted) more nearly than the line. The points arranged about the line PQ depart as far from that line as do the points above the higher line, but they do not depart systematically, as if tending to lie along a smooth curve. When points arrange themselves as at the top of Fig. 104, one must infer that the relation connecting the given data is not exponential in character.

ceed to determine the equation of this line as approximately representing the relation between x and y.

It is easy to find the equation of such a line referred to the uniform scales AB and BC of form $M5$. We may imagine that all rulings are erased and replaced by extensions of the uniform AB scale, as in Fig. 101. The equation of the line PQ is then

$$Y = mX + B \tag{1}$$

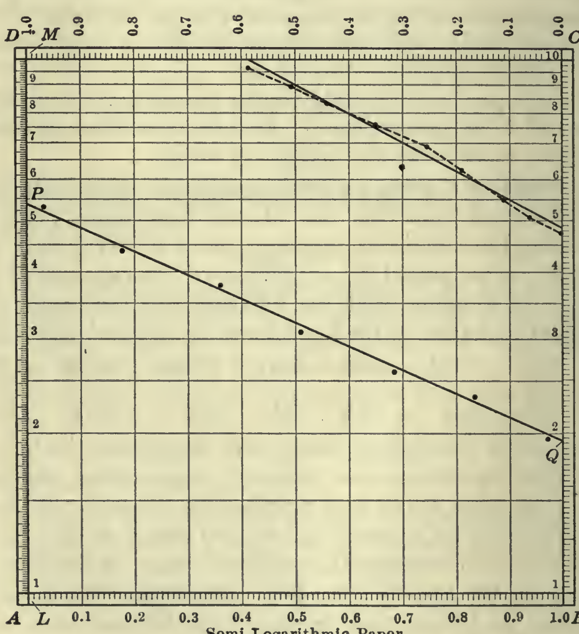


Fig. 104.—Empirical Equations Determined by Use of Form $M5$.

where m is the slope, and B is the y-intercept. Now, for PQ, $m = -0.447$ and $B = 0.730 = \log 5.37$. Equation (1) of PQ becomes, therefore:

$$Y = -0.447X + 0.730$$

or, replacing Y by log y and 0.730 by log 5.37, in order to refer the curve to the scales AB and LM,

$$\log y - \log 5.39 = -0.447x$$

whence

$$y = 5.39(10^{-0.447x}) \tag{2}$$

If it is desired to express the relation to the base e instead of base 10, we may note $10 = e^{2.3026}$ (**§143**, equation (1)), or, substituting in (2),

$$\begin{aligned} y &= 5.39\ (e^{2.303})^{-0.447x} \\ &= 5.39\ e^{-1.029x} \end{aligned} \tag{3}$$

The same result might have been obtained directly by use of the uniform scale AD, at the left of form $M5$. This scale is so constructed that the length 1 on AB corresponds to the length 2.3026 on AD. Now, we know that $e^{2.3026} = 10$, hence we may replace 10 in 10^{mx} by e if we make m in 10^{mx} 2.3026 times as great as before. This is readily done by measuring the slope of PQ by the use of the uniform scale AD instead of the uniform scale BC. Computing the slope of PQ by use of the scale AD we find:

$$\begin{aligned} Y \text{ of } Q &= 0.653 \\ Y \text{ of } P &= \underline{1.681} \\ \text{Difference} &= -1.028 \end{aligned}$$

Since $AB = 1$, this is the slope of the line, measured to the scale AD, and is therefore the value of m in the equation

$$y = ae^{mx} \tag{4}$$

Hence the equation of PQ is

$$y = 5.39e^{-1.028x}$$

which agrees with the equation previously obtained.

154. Change of Scale on Semi-logarithmic Paper. A sheet of semi-logarithmic paper, form $M5$, is a square. If sheets of this paper be arranged "checker-board fashion" over the plane, then the vertical non-uniform scale will be a repetition of the scale LM, Fig. 104, except that the successive segments of length LM must be numbered 1, 2, 3, . . ., 9 for the original LM, then 10, 20, 30, . . ., 90 for the next vertical segment of the checker-board, then 100, 200, 300, . . ., 900, for the next, etc. It is obvious, therefore, that the initial point A of a sheet of semi-logarithmic paper may be said to have the ordinate 1, or 10, or 100, etc., or 10^{-1}, 10^{-2}, etc., as may be most convenient for the particular graph under consideration. The horizontal scale being a uniform scale, any values of x may be plotted to any convenient scale on it, as when using ordinary squared paper. However, if the horizontal unit of length (the length AB, form $M5$) be taken as any value different from unity, then the slope m of the line PQ drawn on the semi-logarithmic paper can only be found by dividing its **apparent slope** by the scale value of the side AB. That is, the correct value of m in

$$y = a10^{mx}$$

is, in all cases,

$$m = \frac{\text{apparent slope of } PQ}{\text{scale value of } AB}$$

The "apparent slope" of PQ is to be measured by applying any convenient uniform scale of inches, centimeters, etc., to the horizontal and vertical sides of a right triangle of which PQ is the hypotenuse.

Exercises

1. A thermometer bulb initially at temperature 19.3° C. is exposed to the air and its temperature θ noted at various times t (in seconds) as follows:

t	0	20	40	60	80	100	120
θ	19.3	14.2	10.4	7.6	5.6	4.1	3.0

Plot these results on semi-logarithmic paper and test whether or not θ follows the compound interest law. If so, determine the value of θ_0 and b in the equation $\theta = \theta_0 e^{-bt}$. Note that the last point given by the table, namely $t = 120$, $\theta = 3.0$, goes into a new square if the scale AB be called 0—100. If the scale AB be called 0—200 then all entries can appear on a single sheet of form $M5$.

2. Graph the following on semi-logarithmic paper:

n	1/2	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
w	1.6	3.0	5.1	8.0	14.0	23.0

Show that the equation connecting n and w is $w = 10^{0.45n}$.

Suggestion: The scale AB, form $M5$, may be called 0—5 for the purpose of graphing n.

5. Graph the following on semi-logarithmic paper, and find the equation connecting n and w.

n	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
w	2.60	3.41	4.45	5.75	7.56	9.85	1.30	16.6

4. A circular disk is suspended by a fine wire at its center. When at rest the upper end of the wire is turned by means of a supporting knob through 30°. The successive angles of the torsional swings of the disk from the neutral point are then read at the end of each swing as follows:

Swing number	1	2	3	4	5	6	7
Angle	26°.4	23°.2	20°.5	18°.0	15°.9	14°.0	12°.3

Show that the angle of the successive swings follows the compound interest law and find in at least two different ways the equation connecting the number of the swing and the angle. Show by the slide rule that the compound interest law holds.

155. The Power Function Compared with the Exponential Function. It has been emphasized in this book that the fundamental laws of natural science are three in number, namely: (1) the parabolic law, expressed by the power function $y = ax^n$ where n may be either positive or negative; (2) the harmonic or periodic law, $y = a \sin nx$, which is fundamental to all periodically occurring phenomena; and (3) the compound interest law discussed in this chapter. While there are other important laws and functions in mathematics, they are secondary to those expressed by these fundamental functions. The second of the functions above named will be more fully discussed in the chapter on waves. The discussion of the compound interest law should not be closed without a careful comparison of power functions and exponential functions.

The characteristic property of the power function

$$y = ax^n \tag{1}$$

is that *as x changes by a constant factor, y changes by a constant factor also.* Take

$$y = ax^n = f(x) \tag{2}$$

Let x change by a constant factor m, so that the new value of x is mx. Call y' the new value of y. Then

$$y' = a(mx)^n = f(mx) \tag{3}$$

That is:

$$\frac{y}{y'} = \frac{a(mx)^n}{ax^n} = m^n \tag{4}$$

which shows that *the ratio of the two* y*'s* is independent of the value of x used, or *is constant for constant values of* m.

Another statement of the law of the power function is: *As* x *increases in geometrical progression,* y*, or the power function, increases in geometrical progression also.*

Let m be nearly 1, say $1 + r$, where r is the percent change in x and is small, then we have:

$$\frac{y'}{y} = \frac{f(x + rx)}{f(x)} = \frac{a(x + rx)^n}{ax^n} = (1 + r)^n = 1 + nr \tag{5}$$

by the approximate formula for the binomial theorem (§**111**).

Hence, replacing 1 on the right side of (5) by $\frac{f(x)}{f(x)}$ and transposing:

$$\frac{y' - y}{y} = \frac{f(x + rx) - f(x)}{f(x)} = nr \tag{6}$$

The fraction in the first member is the percent change in y *or in* $f(x)$. The number r is the percent change in the variable x. Therefore (6) states that for small changes of the variable the *percent of change in the function is* n *times the percent of change in the variable.*

Let the exponential function be represented by

$$y = ae^{bx} = F(x) \tag{7}$$

As already noted in the preceding sections, *increasing* x *by a constant term increases* y *or the function by a constant factor.* Thus

$$\frac{y'}{y} = \frac{F(x + h)}{F(x)} = \frac{ae^{b(x+h)}}{ae^{bx}} = e^{bh} \tag{8}$$

which is independent of the value of x or is *constant for constant* h. The expression e^{bh} is the *factor* by which y or the function is increased when x is increased by the *term* or *increment* h. See also §**147** and Fig. 97.

In other words, as x increases in arithmetical progression, y

or the exponential function increases in geometrical progression.

The percent of change is:

$$\frac{F(x + h) - F(x)}{F(x)} = e^{bh} - 1 \tag{9}$$

which is constant for constant increments h *added* to the variable x.

If x change by a constant percent from x to $x(1 + r)$, it will be found that the percent change in the function is not constant, but is variable.

The above properties enable one to determine whether measurements taken in the laboratory can be expressed by functions of either of the types discussed; if the numerical data satisfy the test that if the argument change by a constant factor the function also changes by a constant factor, then the relation may be represented by a power function. If, however, it is found that a change of the argument by a constant increment changes the function by a constant factor, then the relation can be expressed by an equation of the exponential type.

We have already shown how to determine the constants of the exponential equation by graphing the data upon semi-logarithmic paper. In case the equation representing the function is of the form:

$$y = ae^{bx} + c \tag{10}$$

then the curve is not a straight line upon semi-logarithmic paper. If tabulated observations satisfy the condition that the function less (or plus) a certain constant increases by a constant factor as the argument increases by a constant term, then the equation of the type (10) represents the function and the other constants can readily be determined.

The determination of the equations of curves of the parabolic and hyperbolic type is best made by plotting the observed data upon logarithmic coördinate paper as explained in the next section.

156. Logarithmic Coördinate Paper. If coördinate paper be prepared on which the uniform x and y scales are both replaced by non-uniform scales divided proportionately to $\log x$ and $\log y$ respectively, then it is possible to show that *any curve of the parabolic or hyperbolic type when drawn upon such coördinate paper will* *be a straight line.* This kind of squared paper is called **logarithmic paper,** and is illustrated in Fig. 105.

To find the equation of a line PQ on such paper, we imagine, as in the case of semi-logarithmic paper, that all rulings are erased and replaced by continuations of the uniform scales ON and MN, on which the length ON or MN is taken as unity. Denoting, as

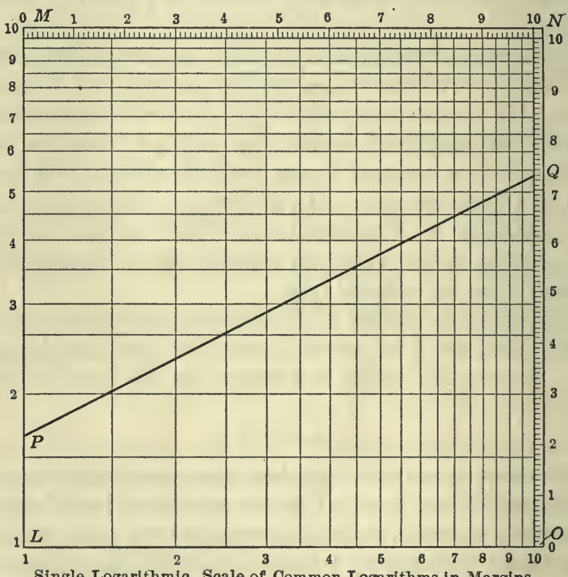


FIG. 105.—Logarithmic Coördinate Paper, Form $M4$. The finer rulings of form $M4$ are not reproduced.

before, distances referred to these uniform scales by capital letters, we may write as the general equation of a straight line:

$$Y = mX + B \tag{1}$$

In the case of the line PQ, $m = 0.505$, $B = 0.219$, and hence

$$Y = 0.505X + 0.219$$

But, $Y = \log y$, $X = \log x$, where y and x represent distances measured on the scales LM and LO respectively, and $0.219 = \log 1.65$. Hence:

$$\log y = 0.505 \log x + \log 1.65$$

or,

$$\log y - \log 1.65 = 0.505 \log x$$

This may be written in the form

$$\log \frac{y}{1.65} = \log x^{0.505}$$

whence

$$\frac{y}{1.65} = x^{0.505}$$

or,

$$y = 1.65 x^{0.505} \tag{2}$$

In general, if $B = \log b$, we may write the equation (1) in the form

$$y = bx^m \tag{3}$$

If the straight line on logarithmic paper passes through the point (1, 1) its Cartesian equation is

$$Y = mX \tag{4}$$

or, referred to the logarithmic scales,

$$\log y = m \log x$$

or,

$$y = x^m. \tag{5}$$

If the straight line on logarithmic paper passes through the point (a, b) with slope m, its equation referred to the logarithmic scales is

$$\frac{y}{b} = \left[\frac{x}{a}\right]^m \tag{6}$$

On logarithmic paper, form $M4$, the numbers printed in the lower and in the left margin refer to the non-uniform scale in the body of the paper. By calling the left-hand lower corner the point (1, 10), (10, 10), (10, 1), (10, 100), (1, 100) or (100, 100), . . ., instead of (1, 1), these numbers may be changed to 10, 20, 30, . . . or to 100, 200, 300, . . ., etc.

In the following exercises the graphs are to be carefully constructed upon logarithmic paper, and the values of the various graduations and all other necessary information indicated on the paper in terms of the proper concrete units.

If the range of any variable is to extend beyond any of the single decimal intervals, 1—10, 10—100, 100—1000, . . ., the "multiple paper," form $M6$, may be used, or several straight lines may be drawn across form $M4$ corresponding to the value of the function in each decimal interval, 1—10, 10—100, . . ., so that as many straight lines will be required to represent the function on the first sheet as there are intervals of the decimal scale to be represented. However, if the exponent m in $y = bx^m$ be a rational number, say n/r, then the lines required for all decimal intervals will reduce to r different straight lines.

One of the most important uses of logarithmic paper is the determination of the equation of a curve satisfied by laboratory data. If such data, when plotted on logarithmic paper, appear as a straight line, an equation of the parabolic type satisfies the observations and the equation is readily found. The exponent m is determined by measuring the slope of the line with an ordinary uniform scale. The equation of the line is best found by noting the coördinates of any one point (a, b) and substituting these and the slope m in the equation

$$\frac{y}{b} = \left[\frac{x}{a}\right]^m.$$

Exercises

Draw the following on single or multiple logarithmic paper, forms $M4$ or $M6$:

1. $y = x,\ y = 2x,\ y = 3x,\ y = 4x,\ . . .$

2. $y = x,\ y = x^2,\ y = x^3,\ y = x^4,\ . . .$

3. $y = 1/x,\ y = 1/x^2,\ y = 1/x^3,\ . . .$

4. $y = x^{1/2},\ y = x^{1/3},\ y = x^{2/3},\ . . .$

5. $A = \pi r^2$.

6. $p = 0.003v^2$, where p is the pressure in pounds per square foot on a flat surface exposed to a wind velocity of v miles per hour.

7. $v = c\sqrt{rs}$ for $c = 110$ and $r = 1$.

8. $f = \sqrt{2gh}$ for $g = 32.2$.

9. $C = E/R$ where $E = 110$ volts.

10. $s = (1/2)gt^2$ where $g = 32.2$.

11. $T = \pi\sqrt{L/g}$, where $g = 32.2$.

12. $p/p_0 = (\rho/\rho_0)^{1.408}$, where $\rho_0 = 0.075$, the weight of 1 cubic foot of air in pounds at 70° F. and at pressure p_0 of 14.7 pounds per square inch.

13. $H = \dfrac{S^3 D^{2/3}}{C}$, for $D = 5000$, 10,000, 15,000 20,000, where $C =$ 225, D is displacement in tons and S is speed in knots.

14. $H = \dfrac{d^3 N}{50}$, for $N = 100$, 200, 300, 400, 500, 600, 700, 800, 900, 1000. d is the diameter of cold rolled shafting in inches; the line should be graphed for values of d between $d = 1$ and $d = 10$.

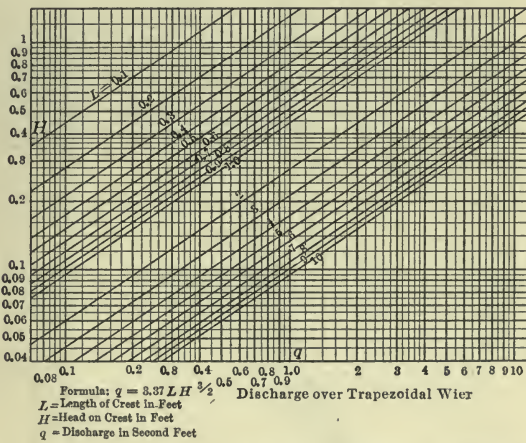


FIG. 106.—A Weir Formula Graphed on Multiple Logarithmic Paper.

15. $F = 0.000341WRN^2$, where N is revolutions per minute, R is radius in feet, W is weight in pounds, and F is centrifugal force in pounds.

16. $q = 3.37Lh^{3/2}$ for $L = 0.1$, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. See Fig. 106.

17. $H = \dfrac{0.38\,V^{1.86}}{d^{1.25}}$, where V is the velocity of water in feet per second under the head of H feet per 10,000 feet in clean cast-iron pipe of diameter d feet. See Fig. 108.

18. The relation between electrical resistance and amount of total solids in solution for Arkansas River Valley water at 70° F. is given by the following table:

S = total solids in solution as parts per 1,000,000:
1,000, 800, 600, 400, 300, 200
R = resistance in ohms: 215, 260, 340, 480, 615, 860.

Plot the results on form $M4$ and find from the graph the equation connecting S and R.

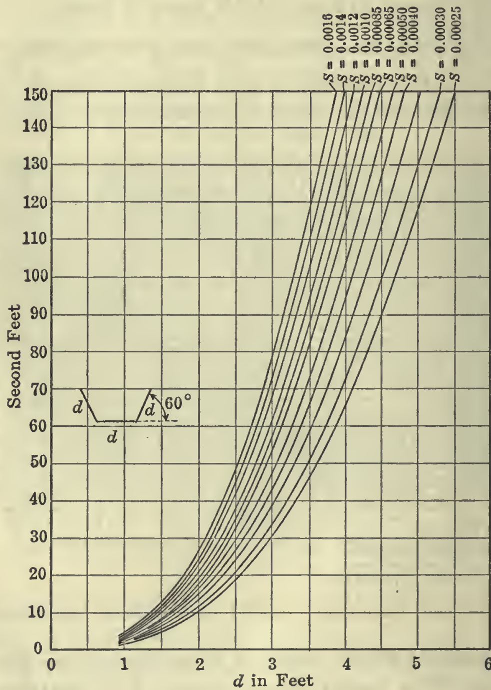


Fig. 107.—Capacity in Cubic Feet per Second of Trapezoidal Smooth Concrete Flumes for Various Gradients (S) and for Various Dimensions (d).

19. Replot the curves of Fig. 107. On the new diagram draw the lines corresponding to slopes of 7, 8, and 9 feet per 10,000 respectively.

20. Explain the periodic character of the rulings on Figs. 106 and 108.

157. Sums of Exponential Functions. Functions consisting of the sum of two different exponential functions are of frequent occurrence in the application of mathematics, especially in electrical science. Types of fundamental importance are $e^u + e^{-u}$ and $e^u - e^{-u}$ which are so important that the forms $(e^u + e^{-u})/2$ and $(e^u - e^{-u})/2$ have been given special names and tables of their values have been computed and printed. The first of

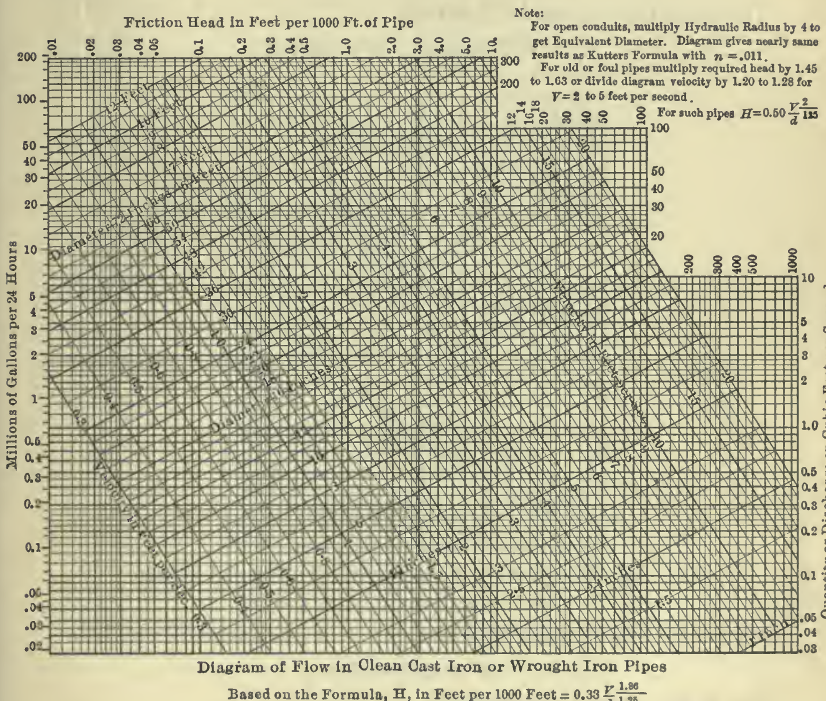


FIG. 108.—A Complicated Example of the Use of Multiple Logarithmic Paper, Form $M6$. From Transactions Am. Soc. C. E. Vol. LI.

these is called the **hyperbolic cosine of u** and the second is called the **hyperbolic sine of u;** they are written in the following notation:

$$\cosh u = (e^u + e^{-u})/2, \quad \sinh u = (e^u - e^{-u})/2$$

If $x = a \cosh u$ and $y = a \sinh u$, then squaring and subtracting

$$x^2 - y^2 = a^2(\cosh^2 u - \sinh^2 u)$$
$$= a^2\left[\frac{e^{2u} + 2 + e^{-2u}}{4} - \frac{e^{2u} - 2 + e^{-2u}}{4}\right]$$
$$= a^2$$

Therefore the hyperbolic functions

$$x = a \cosh u,\ y = a \sinh u$$

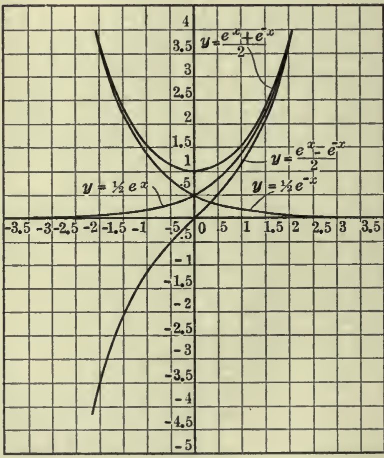


FIG. 109.—The Curves of the Hyperbolic Sine and Cosine.

appear in the parametric equations of a rectangular hyperbola

$$x^2 - y^2 = a^2,$$

just as the circular functions

$$x = a \cos \theta,\ y = a \sin \theta$$

appear in the parametric equations of the circle

$$x^2 + y^2 = a^2$$

The graphs of $y = a \cosh x$ and $y = a \sinh x$ were called for in exercises 1, 2, **§146.** They are shown in Fig. 109. The first of these curves is formed when a chain is suspended between two points of support; it is called the **catenary.** These two curves are best drawn by averaging the ordinates of $y = e^x$ and $y = e^{-x}$, and the ordinates of $y = e^x$ and $y = -e^{-x}$.

Curves whose equations are of the form $y = ae^{mx} + be^{nx}$ take on quite a variety of forms for various values of the constants. A good idea of certain important types can be had by a comparison of the curves of Fig. 110 whose equations are:

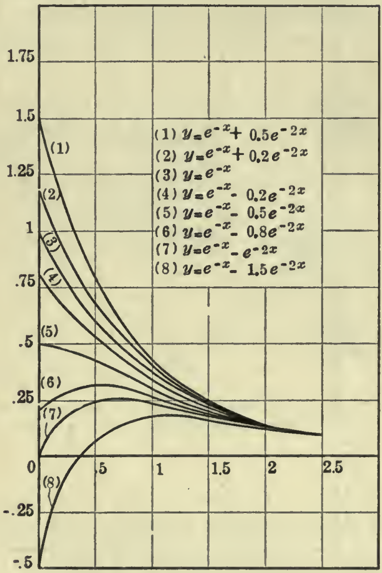


FIG. 110.—Combinations of Two Exponential Curves. After Steinmetz.

$$
\begin{aligned}
y &= e^{-x} + 0.5e^{-2x}\\
y &= e^{-x} + 0.2e^{-2x}\\
y &= e^{-x}\\
y &= e^{-x} - 0.2e^{-2x}\\
y &= e^{-x} - 0.5e^{-2x}\\
y &= e^{-x} - 0.8e^{-2x}\\
y &= e^{-x} - e^{-2x}\\
y &= e^{-x} - 1.5e^{-2x}
\end{aligned}
$$

The student should arrange in tabular form the necessary numerical work for the construction of these curves.

If the second exponent be increased in absolute value, the points of intersection with the y-axis remain the same, but the region of close approach of the curves to each other is moved along the curve $y = e^{-x}$ to a point much nearer the y-axis. To show this the following curves have been drawn and shown in Fig. 111.

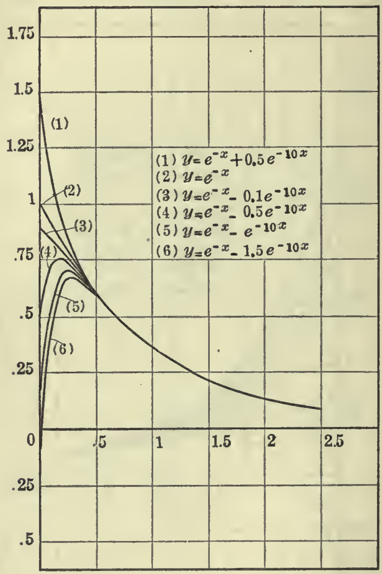


FIG. 111.—Combinations of Two Exponential Curves. After Steinmetz.

$$
\begin{aligned}
y &= e^{-x} + 0.5e^{-10x}\\
y &= e^{-x}\\
y &= e^{-x} - 0.1e^{-10x}\\
y &= e^{-x} - 0.5e^{-10x}\\
y &= e^{-x} - e^{-10x}\\
y &= e^{-x} - 1.5e^{-10x}.
\end{aligned}
$$

158.* Damped Vibrations. If a body vibrates in a medium like a gas or liquid, the amplitude of the swings are found to get smaller and smaller, or the motion slowly (or rapidly in some cases) dies out. In the case of a pendulum vibrating in oil, the rate of decay of the amplitude of the swings is rapid, but the ordinary rate of the decay of such vibrations in air is quite slow. The ratio between the lengths of the successive amplitudes of vibration is called the **damping factor** or the **modulus of decay.**

The same fact is noted in case the vibrations are the torsional vibrations of a body suspended by a fine wire or thread. Thus a **viscometer,** an instrument used for determining the viscosity of lubricating oils, provides means for determining the rate of the decay of the torsional vibration of a disk, or of a circular cylinder

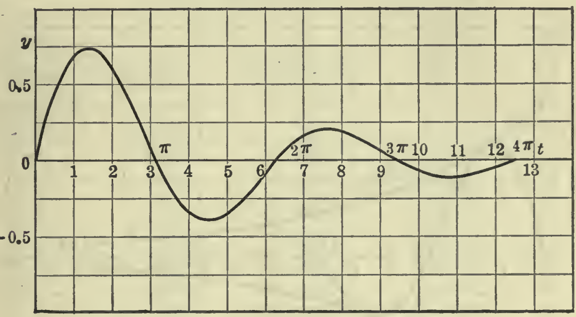


FIG. 112.—The Curve $y = e^{-t/5} \sin t$.

suspended in the oil by a fine wire. The "amplitude of swing" is in this case the angle through which the disk or cylinder turns, measured from its neutral position to the end of each swing.

In all such cases it is found that the logarithms of the successive amplitudes of the swings *differ by a certain constant amount* or, as it is said, the *logarithmic decrement is constant.* Therefore the amplitudes must satisfy an equation of the form

$$A = ae^{-bt}$$

where A is amplitude and t is time. The actual motion is given by an equation of the form

$$y = ae^{-bt} \sin ct$$

A study of oscillations of this type will be more fully taken up in the calculus, for the present it will suffice to graph a few examples of this type. Let the expression be

$$y = e^{-t/5} \sin t \tag{1}$$

A table of values of t and y must first be derived. There are three ways of proceeding: (1) Assign successive values to t irrespective of the period of the sine (see Table V and Fig. 112). (2) Select for the values of t those values that give aliquot parts of the period 2π of the sine (see Table VI and Fig. 113). (3) Draw the sinusoid $y = \sin t$ carefully to scale by the method of §**55**; then draw upon the same coördinate axes, using the same units of measure

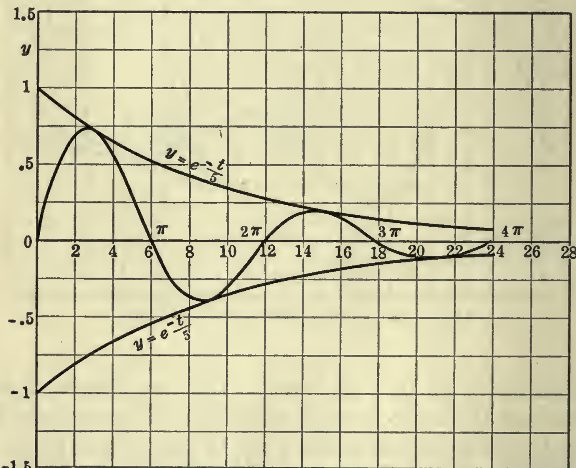


FIG. 113.—The Curve $y = e^{-t/5} \sin t$.

adopted for the sinusoid, the exponential curve $y = e^{-t/5}$; finally multiply together, on the slide rule, corresponding ordinates taken from the two curves, and locate the points thus determined.

The first method involves very much more work than the second for two principal reasons: First, tables of the logarithms of the trigonometric functions with the radian and the decimal divisions of the radian as argument are not available; for this reason 57.3° must be multiplied by the value of t in each case so that an ordinary trigonometric table may be used; second, each of the values written in column (3) of the table must be separately determined, while if the periodic character of the sine be taken advantage of, the numerical values would be the same in each quadrant.

The second method, because of the use of aliquot divisions of the period of the sine, such as $\pi/6$ or $\pi/12$ or $\pi/18$ or $\pi/20$, etc., possesses the advantage that the values used in column (3) need be found for one quadrant only and the values required in column (2) are quite as readily found on the slide rule as in the first method.

TABLE V

Table of the function $y = e^{-t/5} \sin t$

1	2	3	4	5
t in radians	$\log e^{-t/5} = -(0.0869)t$	log sin t or log sin $57.3t$ if t is in degrees	log y	y
0.0	− 0.0000	....	....	+ 0.000
0.5	− 0.0434	9.6807	9.6372	+ 0.434
1.0	− 0.0869	9.9250	9.8381	+ 0.689
1.5	− 0.1303	9.9989	9.8686	+ 0.739
2.0	− 0.1737	9.9587	9.7850	+ 0.610
2.5	− 0.2172	9.7771	9.5599	+ 0.363
3.0	− 0.2606	9.1498	8.8892	+ 0.077
3.5	− 0.3040	9.5450	9.2410	− 0.174
4.0	− 0.3474	9.8790	9.5312	− 0.340
4.5	− 0.3909	9.9901	9.5992	− 0.397
5.0	− 0.4343	9.9818	9.5475	− 0.353
5.5	− 0.4777	9.8485	9.3708	− 0.235
6.0	− 0.5212	9.4464	8.9252	− 0.084
6.5	− 0.5646	9.3322	8.7679	+ 0.059
7.0	− 0.6080	9.8175	9.2095	+ 0.162
7.5	− 0.6515	9.9722	9.3207	+ 0.209
8.0	− 0.6949	9.9954	9.3005	+ 0.200
8.5	− 0.7383	9.9022	9.1634	+ 0.146
9.0	− 0.7817	9.6149	8.8332	+ 0.068
9.5	− 0.8252	8.8760	8.0508	− 0.011
10.0	− 0.8686	9.7356	8.8670	− 0.074
10.5	− 0.9120	9.9443	9.0323	− 0.108
11.0	− 0.9555	9.9999	9.0444	− 0.111
11.5	− 0.9989	9.9422	8.9433	− 0.088
12.0	− 1.0424	9.7296	8.6872	− 0.049

TABLE VI

Table of the function $y = e^{-t/5} \sin t$

1	2	3	4	5
$n = t$ in units of $\pi/6$ radians	$\log e^{-n\pi/30} = -(0.0455)n$	$\log \sin n\pi/6$	$\log y$	y
0	− 0.0000	....	....	0.000
1	− 0.0455	9.6990	9.6535	+ 0.450
2	− 0.0910	9.9375	9.8465	+ 0.702
3	− 0.1364	0.0000	9.8636	+ 0.731
4	− 0.1819	9.9375	9.7556	+ 0.570
5	− 0.2274	9.6990	9.4716	+ 0.296
6	− 0.2729	....	....	+ 0.000
7	− 0.3184	9.6990	9.3806	− 0.240
8	− 0.3638	9.9375	9.5737	− 0.375
9	− 0.4093	0.0000	9.5907	− 0.390
10	− 0.4548	9.9375	9.4827	− 0.304
11	− 0.5003	9.6990	9.1987	− 0.158
12	− 0.5458	....	....	0.000
13	− 0.5912	9.6990	9.1078	+ 0.128
14	− 0.6367	9.9375	9.3008	+ 0.200
15	− 0.6822	0.0000	9.3178	+ 0.208
16	− 0.7277	9.9375	9.2098	+ 0.162
17	− 0.7732	9.6990	8.9258	+ 0.084
18	− 0.8186	....	....	0.000
19	− 0.8641	9.6990	8.8349	− 0.068
20	− 0.9016	9.9375	9.0279	− 0.107
21	− 0.9551	0.0000	9.0449	− 0.111
22	− 1.0006	9.9375	8.9369	− 0.087
23	− 1.0460	9.6990	8.6530	− 0.045
24	− 1.0915	....	....	0.000

The third method is perhaps more desirable than either of the others if more than two figures accuracy is not required. The curve can readily be drawn with the scale units the same in both dimensions, as is sometimes highly desirable in scientific applications.

In Figs. 112 and 113 a larger unit has been used on the vertical scale than on the horizontal scale. In Fig. 113 the horizontal unit is incommensurable with the vertical unit. To draw the curve to a true scale in both dimensions it is preferable to lay off the coördinates on plain drawing paper and not on ordinary squared paper. Rectangular coördinate paper is not adapted to the proper construction and discussion of the sinusoid, or of curves, like the present one, that are derived therefrom.

Curves whose equations are of the form $y = \frac{1}{2}e^{-t/5} \sin t$ or $y = 3e^{-t/5} \sin t$, etc., are readily constructed, since the constants 1/2, 3, etc., merely multiply the ordinates of (1) by 1/2, 3, etc., as the case may be. Likewise the curve $y = e^{-bx} \sin cx$ is readily drawn since $\sin cx$ can be made from $\sin x$ by multiplying all abscissas of $\sin x$ by $1/c$.

CHAPTER IX

TRIGONOMETRIC EQUATIONS AND THE SOLUTION OF TRIANGLES

A. FURTHER TRIGONOMETRIC IDENTITIES

159. Proof that $\rho = a \cos \theta + b \sin \theta$ is a Circle. *I. Geometrical Explanation.* We know (§**64**) that $\rho_1 = a \cos \theta$ is the polar equation of a circle of diameter a, the diameter coinciding in direction with the polar axis OX; for example, the circle OA, Fig. 114. Likewise, $\rho_2 = b \sin \theta$ is a circle whose diameter is of length b and makes an angle of $+ 90°$ with the polar axis OX, as the circle OB, Fig. 114. Also, $\rho = c \cos (\theta - \theta_1)$ is a circle whose diameter c has the direction angle θ_1. See equation (4), §**68**. We shall show that if the radii vectores corresponding to any value of θ in the equations $\rho_1 = a \cos \theta$ and $\rho_2 = b \sin \theta$ be added together to make a new radius vector ρ, then, for all values of θ, the extremity of ρ lies on a circle (the circle OC, Fig. 114) of diameter $\sqrt{a^2 + b^2}$. In other words we shall show that:

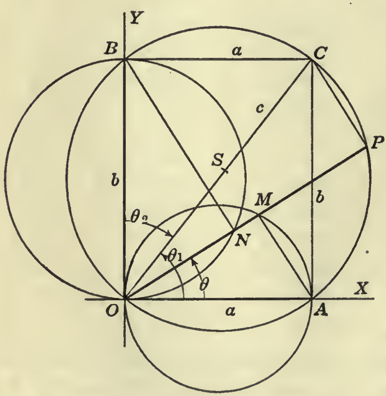


Fig. 114.—Combination of the Circles $\rho = a \cos \theta$ and $\rho = b \sin \theta$ into a Single Circle $\rho = a \cos \theta + b \sin \theta$.

$$\rho = a \cos \theta + b \sin \theta \tag{1}$$

is the equation of a circle.

In Fig. 114, $\rho_1 = a \cos \theta$ will be called the *a-circle* OA; $\rho_2 = b \sin \theta$ will be called the *b-circle* OB. For any value of the angle θ draw radii vectores OM, ON, meeting the a- and b-circles respectively at the points M and N. If P be the point of intersection of MN produced with the circle whose diameter is the diagonal OC of the rectangle described on OA and OB, we shall show that $OM + ON = OP$, no matter in what direction OP be drawn.

Let the circle last mentioned be drawn, and project BC on OP. Since ONB and OPC are right angles, NP is the projection of BC $(= a)$ upon OP. But OM also is the projection of a $(= OA)$ upon OP. Hence $NP = OM$ because the projections of equal parallel lines on the same line are equal. Therefore, for all values of θ, $NP = \rho_1$ and $OP = ON + NP = \rho_2 + \rho_1$, which is the fact that was to be proved.

Designating the angle AOC by θ_1, the equation of the circle OC is by §68

$$\rho = \sqrt{a^2 + b^2} \cos (\theta - \theta_1) \tag{2}$$

The value of θ_1 is known, for its tangent is $\frac{b}{a}$. It should be observed that there is no restriction on the value of θ. As the point P moves on the circle OC, the circumference is twice described as θ varies from 0° to 360°, but the diagram for other positions of the point P is in no case essentially different from Fig. 114.

The above reasoning and the diagram involve the restriction that both a and b are positive numbers. While it is possible to supplement the reasoning to cover the cases in which this restriction is removed, it will be unnecessary as the analytical proof at the end of this section is applicable for all values of a and b.

Example: From the above we know that the equation $\rho = 6 \cos \theta + 8 \sin \theta$ is a circle. The diameter of the circle is $\sqrt{a^2 + b^2} = \sqrt{6^2 + 8^2} = 10$, so that the equation of the circle may also be written in the form $\rho = 10 \cos (\theta - \theta_1)$, in which θ_1 is the angle whose tangent is $\frac{b}{a} = \frac{8}{6} = 1.33$. From a table of tangents $\theta_1 = 53° \ 8'$, so that the equation of the circle may be written $\rho = 10 \cos (\theta - 53° \ 8')$.

II. Analytical Proof. We shall prove analytically that $\rho =$

$a \cos \theta + b \sin \theta$ is a circle, without imposing conditions upon the algebraic signs of a and b. Multiply both members of

$$\rho = a \cos \theta + b \sin \theta \tag{4}$$

by ρ, and obtain

$$\rho^2 = a\rho \cos \theta + b\rho \sin \theta \tag{5}$$

By **§70,** the expression $\rho \cos \theta$ is the value in polar coördinates of the Cartesian abscissa x; also $\rho \sin \theta$ is the value in polar coördinates of the ordinate y. Likewise $\rho^2 = x^2 + y^2$. After the substitution of these values, (5) becomes

$$x^2 + y^2 = ax + by \tag{6}$$

Transposing and completing the squares:

$$\left[x - \frac{a}{2}\right]^2 + \left[y - \frac{b}{2}\right]^2 = \frac{a^2 + b^2}{4} \tag{7}$$

This is the Cartesian equation of a circle with center at the point $\left[\frac{a}{2}, \frac{b}{2}\right]$, and of radius $\frac{1}{2}\sqrt{a^2 + b^2}$. The circle passes through the origin, since the coördinates (0, 0) satisfy the equation, and also passes through the point (a, b) since these coördinates satisfy (6).

Since (4) is now known to represent a circle passing through the origin, its polar equation can be written in any of the forms (3)–(6) of **§68.** Calling θ_1 the direction angle of the diameter of (7), (no matter what direction OC actually occupies) we can write the eqation of the circle in the form

$$\rho = \sqrt{a^2 + b^2} \cos (\theta - \theta_1) \tag{8}$$

in which the direction angle θ_1 is the angle AOC, Fig. 114, or the angle whose tangent is $b \div a$. If a and b are not both positive, the angle θ_1 is still easily determined. For example, if $a = -1$, and $b = -1$, then θ_1 = angle of third quadrant whose tangent is 1, or $= 225°$, so that equation (8) becomes:

$$\rho = \sqrt{2} \cos (\theta - 225°)$$

This may also be written

$$\rho = \sqrt{2} \cos (\theta + 135°)$$

since the resulting circle may be thought of as $\rho = \sqrt{2} \cos \theta$ rotated negatively through 135°.

The equation of the circle OC in any position, that is, for any values of a and b, positive or negative, may also be written in the form

$$\rho = \sqrt{a^2 + b^2} \sin (\theta + \theta_2) \tag{9}$$

in which θ_2 is the angle BOC in Fig. 114. See §**68,** equations (5) and (6).

It has been emphasized above that θ, θ_1, θ_2, are *any* angles—that is, angles not restricted in size or sign. The distinction between them need not be lost sight of, however. θ is *any angle* because it is the *variable* vectorial angle of any point of the locus, and ranges positively and negatively from 0° to any value we please. θ_1 is any angle (positive or negative) because it is the direction angle of the diameter OC. It is a *constant*, but a *general*, or unrestricted, angle, but would usually be taken less than 360° in absolute value. By construction, θ_2 is also any constant angle.

The result of this section should also be interpreted when the variables are x and y in rectangular coördinates, and not ρ and θ of polar coördinates. Thus, $y = a \cos x$ is a sinusoid with highest point or crest at $x = 0, 2\pi, 4\pi, \ldots$ Likewise, $y = b \sin x$ is a sinusoid with crest at $x = \frac{\pi}{2}, \frac{9\pi}{2}, \frac{17\pi}{2}, \ldots$ The above demonstration shows that the curve

$$y = a \cos x + b \sin x$$

is identical with the sinusoid

$$y = \sqrt{a^2 + b^2} \cos (x - h_1) = \sqrt{a^2 + b^2} \sin (x + h_2)$$

of amplitude $\sqrt{a^2 + b^2}$ and with the crest located at $x = h_1$, or at $\frac{\pi}{2} - h_2$, where h_1 is, in radians, the angle whose tangent is $\frac{b}{a}$, and h_2 is, in radians, the angle whose tangent is $\frac{a}{b}$.

Exercises

1. Put the equation $\rho = 2 \cos \theta + 2\sqrt{3} \sin \theta$ in the form $(x - h)^2 + (y - k)^2 = h^2 + k^2$; also in the form $\rho = a \cos (\theta - \theta_1)$ and find the value of θ_1. See equation (7) above.

2. Find the value of θ_1 if $\rho = \cos \theta - \sqrt{3} \sin \theta$.

3. Put the equation $\rho = 4\cos\theta + \frac{4}{3}\sqrt{3}\sin\theta$ in the forms $(x - h)^2 + (y - k)^2 = h^2 + k^2$ and $\rho = a\cos(\theta - \theta_1)$.

4. Put the equation $\rho = -4\cos\theta - 4\sin\theta$ in the form $(x - h)^2 + (y - k)^2 = h^2 + k^2$ and find the value of θ_2 when the given equation is written in the form $\rho = a\sin(\theta + \theta_2)$.

5. Put the equation $\rho = 2\sqrt{3}\cos\theta + 2\sin\theta$ in the form $(x - h^2) + (y - k)^2 = h^2 + k^2$; also in the form $\rho = a\cos(\theta - \theta_1)$.

6. Put the equation $\rho = 3\cos\theta + 4\sin\theta$ in the form $\rho = a\sin(\theta + \theta_2)$. Put the same equation in the form $\rho = a\cos(\theta - \theta_1)$. ($\theta_1$ is the angle AOC, Fig. 114.

7. Put the equation $\rho = 5\cos\theta + 12\sin\theta$ in the form $p = a\sin(\theta + \theta_2)$; also in the form $\rho = a\cos(\theta - \theta_1)$.

8. Put $\rho = 3\cos\theta + 4\sin\theta$ in the form $(x - h)^2 + (y - k)^2 = h^2 + k^2$.

9. Put $\rho = 5\cos\theta + 12\sin\theta$ in the form $(x - h)^2 + (y - k)^2 = h^2 + k^2$.

10. Put the equation $(x - 1)^2 + (y - 1)^2 = 2$ in the form $\rho = a\sin(\theta + \alpha)$ and determine a and α.

11. Put the equation $(x + 1)^2 + (y - \sqrt{3})^2 = 4$ in the form $\rho = a\sin(\theta - \alpha)$ and determine a and α.

12. Put the equation $(x + 1)^2 + (y + \sqrt{3})^2 = 4$ in the form $\rho = a\sin(\theta - \alpha)$ and determine a and α.

13. Put the equation $(x + 1)^2 + (y + 1)^2 = 2$ in the form $\rho = a\cos(\theta + \alpha)$ and determine a and α.

14. Put the equation $(x + 1)^2 + (y + \sqrt{3})^2 = 4$ in the form $\rho = a\cos(\theta + \alpha)$ and determine a and α.

15. Find the maximum value of $\cos\theta - \sqrt{3}\sin\theta$, and determine the value of θ for which the expression is a maximum.

Suggestion: Call the expression ρ. The maximum value of ρ is the diameter of the circle $\rho = \cos\theta - \sqrt{3}\sin\theta$. The direction cosine of the diameter is the value of α when the equation is put in the form $\rho = a\cos(\theta - \alpha)$.

16. Find the value of θ that renders $\rho = \frac{1}{3}\sqrt{3}\cos\theta - \frac{1}{2}\sin\theta$ a maximum and determine the maximum value of ρ.

17. Find the maximum value of $3\cos t + 4\sin t$.

160. Addition Formulas for the Sine and Cosine. From the preceding section, equations (1), (8) and (9), we know that the equation of the circle OC, Fig. 115, may be written in any one of the forms:

$$\rho = a \cos \theta + b \sin \theta \qquad (1)$$

$$\rho = c \sin (\theta - \theta_2) \qquad (2)$$

$$\rho = c \cos (\theta - \theta_1) \qquad (3)$$

Hence, for all values of θ, θ_1, and θ_2,

$$\sin (\theta - \theta_2) = \frac{a}{c} \cos \theta + \frac{b}{c} \sin \theta \qquad (4)$$

$$\cos (\theta - \theta_1) = \frac{a}{c} \cos \theta + \frac{b}{c} \sin \theta \qquad (5)$$

In each of these equations $c = \sqrt{a^2 + b^2}$. The letters a and b stand for the coördinates of C irrespective of their signs or of the position of C.

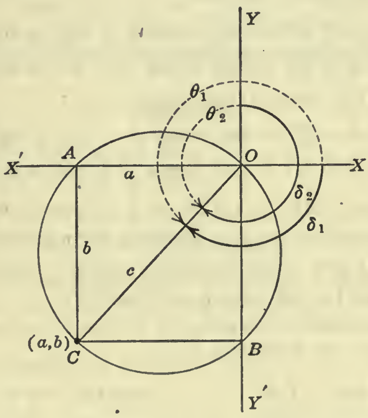


FIG. 115.—The Circle $\rho = c \cos (\theta - \theta_1)$ or $\rho = \sin (\theta - \theta_2,)$ used in the Proof of the Addition Formulas. Note that $\theta_1 = 90° + \theta_2$ which is also true for *negative angles*, namely $\delta_1 = 90° + \delta_2$

Since (4) and (5) are true for all values of θ, they are true when $\theta = 0°$ and when $\theta = 90°$.

First, let $\theta = 0°$ in (4) and (5).

then from (4): $a/c = \sin (- \theta_2) = - \sin \theta_2$ by §**58** (6)

From (5): $a/c = \cos (- \theta_1) = \cos \theta_1$ by §**58** (7)

Second, let $\theta = 90°$ in (4) and (5).

then from (4): $b/c = \sin (90° - \theta_2) = \cos \theta_2$ (8)

From (5): $b/c = \cos (90° - \theta_1) = \sin \theta_1$ (9)

Substituting (6) and (8) in (4); also (7) and (9) in (5), we have

$$\sin (\theta - \theta_2) = \sin \theta \cos \theta_2 - \cos \theta \sin \theta_2 \tag{10}$$

$$\cos (\theta - \theta_1) = \cos \theta \cos \theta_1 + \sin \theta \sin \theta_1 \tag{11}$$

Since these are true for all values of θ_1 and θ_2, put $\theta_1 = (-\epsilon_1)$ and $\theta_2 = (-\epsilon_2)$. Then by §58, (10) and (11) become

$$\sin (\theta + \epsilon_2) = \sin \theta \cos \epsilon_2 + \cos \theta \sin \epsilon_2 \tag{12}$$

$$\cos (\theta + \epsilon_1) = \cos \theta \cos \epsilon_1 - \sin \theta \sin \epsilon_1 \tag{13}$$

To aid in committing these four important formulas to memory, it is best to designate in each case the angles by α and β, and write (12) and (13) in the form

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \tag{14}$$

$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \tag{15}$$

and also write (10) and (11) in the form

$$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \tag{16}$$

$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \tag{17}$$

The four formulas (14), (15), (16) and (17) must be committed to memory. They are called the **addition formulas for the sine and cosine.** The above demonstration shows that the addition formulas are true for all values of α and β.

By the above formulas it is possible to compute the sine and cosine of 75° and 15° from the following data:

$$\sin 30° = 1/2 \qquad \sin 45° = \tfrac{1}{2}\sqrt{2}$$

$$\cos 30° = \tfrac{1}{2}\sqrt{3} \qquad \cos 45° = \tfrac{1}{2}\sqrt{2}$$

Thus:

$$\begin{aligned}\sin 75° = \sin (30° + 45°) &= \sin 30° \cos 45° + \cos 30° \sin 45° \\ &= \tfrac{1}{2}\cdot\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{3}\cdot\tfrac{1}{2}\sqrt{2} \\ &= \tfrac{1}{4}\sqrt{2}\,(\sqrt{3} + 1)\end{aligned}$$

Likewise:

$$\sin 15° = \sin (45° - 30°) = \tfrac{1}{4}\sqrt{2}\,(\sqrt{3} - 1)$$

161. Addition Formula for the Tangent. Dividing the members of (14) §160 by the members of (15) we obtain:

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \tag{1}$$

Dividing numerator and denominator of the last fraction by $\cos\alpha\cos\beta$

$$\tan(\alpha+\beta)=\frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \tag{2}$$

or:

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \tag{3}$$

Likewise it can be shown from (16) and (17), §160, that:

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \tag{4}$$

Equations (3) and (4) are the **addition formulas for the tangent.**

Exercises

1. Compute $\cos 75°$ and $\cos 15°$.

2. Compute $\tan 75°$ and $\tan 15°$.

3. Write in simple form the equation of the circle

$$\rho=\sin\theta+\cos\theta.$$

4. Put the equation of the circle $\rho=3\sin\theta+4\cos\theta$ in the form $\rho=c\sin(\theta+\theta_1)$ and find from the tables, or by the slide rule, the value of θ_1.

5. Derive a formula for $\cot(\alpha+\beta)$.

6. Prove $\cos(s+t)\cos(s-t)=\cos^2 s-\sin^2 t$.

7. Express in the form $c\cos(a-b)$ the binomial $3\cos a+4\sin a$.

8. Express in the form $c\sin(a+b)$ the binomial $5\cos a+12\sin a$.

9. Find the coördinates of the maximum point or crest of the sinusoid $y=\sin x+\sqrt{3}\cos x$. [First reduce the equation to the form $y=c\sin(x+\alpha)$].

10. Prove the addition formulas in the following manner: (1) In $\cos(\theta-\theta_1)=\frac{a}{c}\cos\theta+\frac{b}{c}\sin\theta$, show that $a/c=\cos\theta_1$, $b/c=$

$\sin \theta_1$, for all values of θ_1. (2) Find $\cos(\theta + \theta_2)$ by replacing θ_1 by $(-\theta_2)$. (3) Find $\sin(\phi + \theta_1)$ by the substitution in (1) of $\theta = (\pi/2 - \phi)$. (4) Find $\sin(\theta - \theta_3)$ by replacing θ_1 by $(-\theta_3)$.

162. Functions of Composite Angles. The sine, cosine, or tangent of the angles $(90° - \theta)$, $(90° + \theta)$, $(180° - \theta)$, $(180° + \theta)$, $(270° - \theta)$, $(270° + \theta)$ can be expressed in terms of functions of θ alone by means of the addition formulas of §§**160** and **161.** If θ be an angle of the first quadrant, it is easy, however, to obtain all the relations by drawing the triangles of reference for the various angles and then comparing homologous sides of the similar

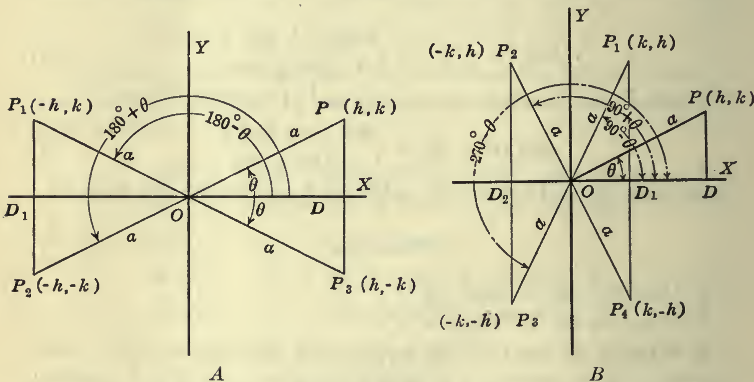


FIG. 116.—An Angle θ Combined with an Even Number of Right Angles, (A) and with an Odd Number of Right Angles, (B).

right triangles of reference. Let the terminal side of the angle θ be OP (Fig. 116 B), and let P be the point (h, k). Let the terminal sides of the angles $(90° - \theta)$, $(90° + \theta)$, $(270° - \theta)$, etc., be cut by the circle of radius a at the points P_1, P_2, P_3, . . . Then the coördinates of P_1 are (k, h); of P_2 are $(-k, h)$; of P_3 are $(-k, -h)$, etc. Hence $\sin \theta = k/a$, $\cos \theta = h/a$, $\sin(90° + \theta) = h/a$, $\cos(90° + \theta) = -k/a$, $\sin(270° - \theta) = -h/a$, $\cos(270° - \theta) = -k/a$, etc., which lead to the equalities:

$$\sin(90° + \theta) = \cos \theta \qquad (1)$$
$$\cos(90° + \theta) = -\sin \theta \qquad (2)$$
$$\sin(270° - \theta) = -\cos \theta \qquad (3)$$
$$\cos(270° - \theta) = -\sin \theta \qquad (4)$$

etc.

By division of (1) by (2) and (3) by (4),

$$\tan(90° + \theta) = -\cot\theta \tag{5}$$

$$\tan(270° - \theta) = \cot\theta \tag{6}$$

Also from Fig. 116 A, $\cos(180° - \theta) = -h/a$, $\sin(180° + \theta) = -k/a$, $\cos(180° + \theta) = -h/a$, $\sin(-\theta) = -k/a$, $\cos(-\theta) = h/a$, whence there results:

$$\sin(180° - \theta) = \sin\theta \tag{7}$$

$$\cos(180° - \theta) = -\cos\theta \tag{8}$$

and by division

$$\tan(180° - \theta) = -\tan\theta \tag{9}$$

also

$$\sin(180° + \theta) = -\sin\theta \tag{10}$$

$$\cos(180° + \theta) = -\cos\theta \tag{11}$$

and by division

$$\tan(180° + \theta) = \tan\theta \tag{12}$$

In the above work the angle θ is drawn as an angle of the first quadrant. The proof that the results hold for all values of θ is best given by means of the addition formulas of §§**160** and **161.** The method will be outlined in the next section.

The results are brought together in the following table. No effort should be made to commit these results to memory in this form. The statements in the form of theorems given below offer a ready means of remembering all of the results.

TABLE VII

Functions of θ Coupled with an Even or with an Odd Number of Right Angles

	$-\theta$	$90°-\theta$	$90°+\theta$	$180°-\theta$	$180°+\theta$	$270°-\theta$	$270°+\theta$
sin	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$
cos	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
tan	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$

For completeness of the table the functions of $(-\theta)$ and of $(90° - \theta)$ have been inserted in columns 1 and 2.

All of the above results can be included in two simple statements. For this purpose it is convenient to separate into different classes the composite angles that are made by coupling θ with an *odd* number of right angles, as $(90° + \theta)$, $(\theta - 90°)$, $(270° - \theta)$, $(450° + \theta)$, etc., and those composite angles that are made by coupling θ with an *even* number of right angles, as $(180° + \theta)$, $(180° - \theta)$, $(360° - \theta)$, $(-\theta)$, etc. Note that 0 is an even number, so that $(-\theta)$ or $(0° - \theta)$ falls into this class of composite angles. We can then make the following statements:

Theorems on Functions of Composite Angles

Think of the original angle θ as an angle of the first quadrant:

I. *Any function of a composite angle made by coupling θ (by addition or subtraction) with an even number of right angles, is equal to the same function of the original angle θ, with an algebraic sign the same as the sign of the function of the composite angle in its quadrant.*

II. *Any function of a composite angle made by coupling θ (by addition or subtraction) with an odd number of right angles, is equal to the co-function of the original angle θ, with an algebraic sign the same as the sign of the function of the composite angle in its quadrant.*

For example, let the original angle be θ, and the composite angle be $(180° + \theta)$. Then any function of $(180° + \theta)$, say $\tan(180° + \theta)$, is equal to $+\tan\theta$, the sign $+$ being the sign of the tangent in the quadrant of the composite angle $(180° + \theta)$ or third quadrant. Likewise $\cot(270° + \theta)$ must equal the negative co-function of the original angle, or $-\tan\theta$, the algebraic sign being the sign of the cotangent in the quadrant of the composite angle $(270° + \theta)$, or fourth quadrant. In the above work it has been assumed that the angle θ is an angle of the first quadrant. The results stated in italics are true, however, no matter in what quadrant θ may actually lie.

163. Functions of Composite Angles. General Proof: All of the results given by Table VII or by theorems I and II above can be deduced at once from the addition formulas, with the especial advantage that the proof holds for all values of the angle θ. Thus, write

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad (1)$$

$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha\sin\beta \qquad (2)$$

Put $\alpha = 180°$, and $\beta = \pm\,\theta$; then (1) and (2) become, respectively:

$$\sin\,(180° \pm \theta) = \mp \sin\,\theta \tag{3}$$
$$\cos\,(180° \pm \theta) = -\cos\,\theta \tag{4}$$

Also in (1) and (2) put $\alpha = 90°$, and $\beta = \pm\,\theta$, then (1) and (2) become, respectively:

$$\sin\,(90° \pm \theta) = \quad \cos\,\theta \tag{5}$$
$$\cos\,(90° \pm \theta) = \mp \sin\,\theta \tag{6}$$

In a similar manner all of the results given in the table may be proved to be true.

164. Angle that a Given Line Makes with Another Line. The slope m of the straight line $y = mx + b$ is the tangent of the

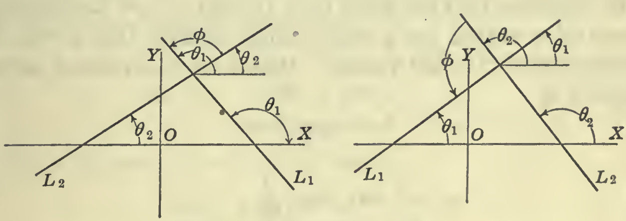


FIG. 117.—The Angle ϕ that a Line L_1 makes with L_2.

direction angle, that is, the tangent of the angle that the line makes with OX. *If L_1 and L_2 are any two lines in the plane, the* **angle that $\mathbf{L_1}$ makes with $\mathbf{L_2}$** *is the positive angle through which L_2 must be rotated about their point of intersection in order that L_2 may coincide with L_1.* Represent the direction angles of two straight lines

$$y = m_1 x + b_1 \tag{1}$$
$$y = m_2 x + b_2 \tag{2}$$

by the symbols θ_1 and θ_2. Then, through the intersection of the lines pass a line parallel to the OX-axis, as shown in Fig. 117. Call ϕ the angle that the line L_1 makes with L_2; that is, the positive angle through which L_2, considered as the initial line, must be turned to coincide with the terminal position given by L_1. If

$\theta_1 > \theta_2$, then $\phi = \theta_1 - \theta_2$, but if $\theta_2 > \theta_1$, then $\phi = 180° - (\theta_2 - \theta_1)$. In either case (by equations (9), **§162,** and (3), **§58**):

$$\tan \phi = \tan (\theta_1 - \theta_2) \tag{3}$$

That is:

$$\tan \phi = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2} \tag{4}$$

or,

$$\mathbf{\tan \phi = \frac{m_1 - m_2}{1 + m_1 m_2}} \tag{5}$$

The condition that the given lines (1) and (2) are parallel is obviously that

$$\mathbf{m_1 = m_2} \tag{6}$$

Thus the lines $y = 5x + 7$ and $y = 5x - 11$ are parallel.

The condition that the given lines (1) and (2) are perpendicular to each other is that $\tan \phi$ shall become infinite; that is, that the denominator of (5) shall vanish. Hence the condition of perpendicularity is

$$1 + m_1 m_2 = 0$$

or,

$$\mathbf{m_1 = -\frac{1}{m_2}} \tag{7}$$

Therefore, *in order that two lines may be perpendicular to each other, the slope of one line must be the negative reciprocal of the slope of the other line.*

Thus the lines $y = (2/3)x - 4$ and $y = -(3/2)x + 2$ are perpendicular.

Exercises

1. Find the tangent of the angle that the first line makes with the second line of each set:

(*a*) $y = 2x + 3$, $\quad y = x + 2$.
(*b*) $y = 3x - 3$, $\quad y = 2x + 1$.
(*c*) $y = 4x + 5$, $\quad y = 3x - 4$.
(*d*) $y = 10x + 1$. $\quad y = 11x - 1$,

2. Find the angle that the first line of each pair makes with the second:

(a) $y = x + 5$,	$y = -x + 5$.
(b) $y = (1/2)x + 6$,	$y = -2x$.
(c) $y = 2x + 4$,	$y = x + 1$.
(d) $2x + 3y = 1$,	$(2/3)x + y = 1$.
(e) $2x + 4y = 3$,	$3x + 6y = 7$.
(f) $2x + 4y = 3$,	$6x - 3y = 7$.

3. Find the angle, in each of the following cases, that the first line makes with the second:

(a) $y = x/\sqrt{3} + 4$,	$y = \sqrt{3}\,x + 2$.
(b) $y = x/\sqrt{3} + 1$,	$y = \sqrt{3}\,x - 4$.
(c) $y = \sqrt{3}\,x - 6$,	$y = \sqrt{3}\,x - 3$.

4. Find the angle that $2y - 6x + 7 = 0$ makes with $y + 2x + 7 = 0$ and also the angle that the second line makes with the first.

165. The Functions of the Double Angle. The addition formulas for the sine, cosine and tangent reduce to formulas of great importance for the special case $\beta = \alpha$.

Thus: $\sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$

or:
$$\sin 2\alpha = 2\sin\alpha\cos\alpha \tag{1}$$

Also: $\cos(\alpha + \alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$

which can be written in the three forms:

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha \tag{2}$$
$$\cos 2\alpha = 2\cos^2\alpha - 1 \tag{3}$$
$$\cos 2\alpha = 1 - 2\sin^2\alpha \tag{4}$$

Forms (3) and (4) are obtained from (2) by substituting, respectively, $\sin^2\alpha = 1 - \cos^2\alpha$ and $\cos^2\alpha = 1 - \sin^2\alpha$.

Equations (3) and (4) are frequently useful in the forms:

$$\cos^2\alpha = \frac{1 + \cos 2\alpha}{2} \tag{5}$$

$$\sin^2\alpha = \frac{1 - \cos 2\alpha}{2} \tag{6}$$

Again:

$$\tan(\alpha + \alpha) = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha\tan\alpha}$$

or:

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} \tag{7}$$

166. The Functions of the Half Angle. From (6) and (5) of §165 we obtain, after replacing α by $u/2$ and extracting the square root,

$$\sin (u/2) = \pm\sqrt{(1 - \cos u)/2} \tag{1}$$

$$\cos (u/2) = \pm\sqrt{(1 + \cos u)/2} \tag{2}$$

Dividing (1) by (2), we obtain:

$$\tan (u/2) = \pm\sqrt{\frac{1 - \cos u}{1 + \cos u}} = \pm\frac{1 - \cos u}{\sin u} = \pm\frac{\sin u}{1 + \cos u} \tag{3}$$

Formulas (1), (2) and (3) have many important applications in mathematics. As a simple example, note that the functions of 15° may be computed when the functions of 30° are known. Thus:

$$\cos 30° = (1/2)\sqrt{3}$$

therefore: $\sin 15° = \sqrt{(1 - \cos 30°)/2} = \sqrt{1/2 - (1/4)\sqrt{3}}$

Also: $\cos 15° = \sqrt{1/2 + (1/4)\sqrt{3}}$

Likewise by (5):

$$\tan 15° = \frac{1 - (1/2)\sqrt{3}}{1/2} = 2 - \sqrt{3}$$

Exercises

1. Compute sin 60° from the sine and cosine of 30°.

2. Compute sine, cosine, and tangent of $22\frac{1}{2}°$.

3. If $\sin x = 2/5$, find the numerical value of $\sin 2x$, and $\cos 2x$ $\tan 2x$, if x be the first quadrant.

4. Show by expanding $\sin (x + 2x)$ that $\sin 3x = 3 \sin x - 4 \sin^3 x$.

5. Prove $\tan 3x = \dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$.

6. Show that $\sin 2\theta/\sin \theta - \cos 2\theta/\cos \theta = 2 \sec \theta$.

7. Show that:

$$\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^2 = 1 + \sin \theta.$$

8. Show that: $\cos 2\theta(1 + \tan 2\theta \tan \theta) = 1$.

9. If $\sin A = 3/5$, calculate $\sin (A/2)$.

10. Prove that $\tan (\pi/4 + \theta) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$.

11. Prove that $\tan (\pi/4 - \theta) = (1 - \tan \theta)/(1 + \tan \theta)$.

12. Show that $\sec \theta + \tan \theta = \dfrac{1 + \sin \theta}{\cos \theta}$.

13. Show that $\dfrac{1 + 2 \sin a \cos a}{\cos^2 a - \sin^2 a} = \dfrac{\cos a + \sin a}{\cos a - \sin a}$.

14. Show that $\sec \theta + \tan \theta = \tan \left[\dfrac{\pi}{4} + \dfrac{\theta}{2}\right]$.

15. Show that $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B$.

16. Prove that $\cos (s + t) \cos (s - t) + \sin (s + t) \sin (s - t) = \cos 2t$.

167. Sums and Differences of Sines and of Cosines Expressed as Products. The following formulas, which permit the substistution of a product for a sum of two sines or of two cosines, are important in many transformations in mathematics, especially in the calculus. They are immediately derivable from the addition formulas; thus, by the addition formulas (14) and (16), **§160,** we obtain:

$$\sin (a + b) + \sin (a - b) = 2 \sin a \cos b$$

Likewise by subtraction of the same formulas:

$$\sin (a + b) - \sin (a - b) = 2 \cos a \sin b$$

By the addition and subtraction, respectively, of the addition formulas for the cosine there results:

$$\cos (a + b) + \cos (a - b) = 2 \cos a \cos b.$$
$$\cos (a + b) - \cos (a - b) = - 2 \sin a \sin b.$$

Represent $(a + b)$ by α and $(a - b)$ by β.
Then $a = (\alpha + \beta)/2$ and $b = (\alpha - \beta)/2$
Hence the above formulas become:

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \tag{1}$$

$$\sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} \tag{2}$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \tag{3}$$

$$\cos \alpha - \cos \beta = - 2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} \tag{4}$$

The principal use of these formulas is in certain transformations in the calculus. A minor use is in adapting certain formulas to logarithmic work by replacing sums and differences by products.

168. Graph of $y = \sin 2x$, $y = \sin nx$, etc. Since the substitution of nx for x in any equation multiplies the abscissas of the curve by $1/n$, or $(n > 1)$ *shortens* or contracts the abscissas of all points of the curve in the uniform ratio $n:1$, the curve $y = \sin 2x$ must have twice as many crests, nodes or troughs in a given interval of x as the sinusoid $y = \sin x$. The curve $y = \sin 2x$ is therefore readily drawn from Fig. 59 as follows: Divide the axis OX into twice as many equal intervals as shown in Fig. 59 and draw vertical lines through the points of division. Then in the new diagram there are twice as many small rectangles as in the original. Starting at O and sketching the diagonals (curved to fit the alignment of the points) of successive cornering rectangles, the curve $y = \sin 2x$ is constructed. It is, of course, the orthographic projection of $y = \sin x$ upon a plane passing through the y-axis and making an angle of 60° (the angle whose cosine is 1/2) with the xy plane. The curve $y = \cos 2x$ is similarly constructed. In each of these cases we see that the period of the function is π and not 2π.

169. Graph of $\rho = \sin 2\theta$, $\rho = \cos 2\theta$, etc. The curve $\rho = \cos\theta$ is the circle of diameter unity coinciding in direction with the axis OX. We have already emphasized that as θ varies from 0° to 360° the *circle is twice drawn*, so that the curve consists of two superimposed circular loops. Now $\rho = \cos 2\theta$ will be found to consist of four loops, somewhat analogous to the leaves of a four-leafed clover, but each loop is described but once as θ varies from 0° to 360°. The curve $\rho = \cos 3\theta$ is a three-looped curve, but each loop is twice drawn as θ varies from 0° to 360°. Also $\rho = \cos 11\theta$ has eleven loops, each twice drawn, while $\rho = \cos 12\theta$ has twenty-four loops, each one described but once, as θ varies from 0° to 360°.

The curves $\rho = \cos 2\theta$, $\rho = \sin 3\theta$, $\rho = \sin\theta/2$ should be drawn by the student upon polar coördinate paper.

By changing the scale of the vectorial angle, the circle of diameter unity may be used as the graph of the equation $\rho = \sin n\theta$. However if two such equations are to be represented at the same time, this expedient is not available, for the vectorial angles of the points of each curve, for the purpose of comparison, must be drawn to a true scale.

170. Graph of $y = \sin^2 x$, $y = \cos^2 x$. The graphs $y = \sin^2 x$ and $y = \cos^2 x$ have important applications in science. The following graphical method offers an easy way of constructing the curves and it illustrates a number of important properties of the functions involved. We shall first construct the curve $y = \cos^2 x$. At the left of a sheet of $8\frac{1}{2} \times 11$-inch paper, draw a circle of radius $\frac{36}{5\pi}$ ($= 2.30$) inches, (OA, Fig. 118). Lay off the angles θ from OA, Fig. 118, as initial line, corresponding to equal intervals (say 10° each) of the quadrant APE as shown in the figure. Let the point P mark any one of these equal intervals. Then dropping the perpendicular AB from A upon OP, the distance OB is the

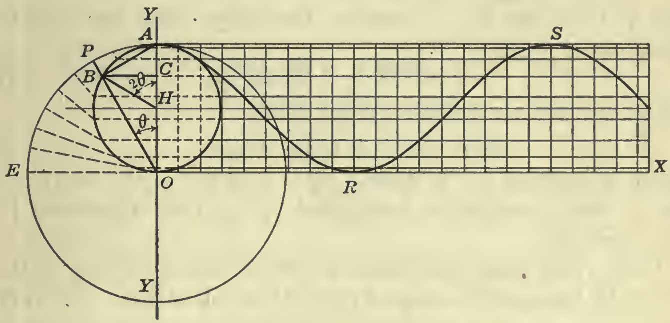


FIG. 118.—The Graph of $y = \cos^2 x$.

cosine of θ, if OA be called unity. Dropping a perpendicular from B upon OA, the distance OC is cut off, which is equal to $\overline{OB}^2$ or $\cos^2 \theta$, since in the right triangle OBA, $\overline{OB}^2 = \overline{OC} \cdot \overline{OA} = \overline{OC} \cdot 1$. Making similar constructions for various values of the angle θ, say for every 10° interval of the arc APE, the line OA is divided at a number of points proportionally to $\cos^2 \theta$. Draw horizontal lines through each point of division of OA. Next divide the axis OX into intervals equal to the intervals of θ laid off on the arc APE; since the radius of the circle OA was taken to be $(36/5\pi)$ inches, an interval of 10° corresponds to an arc of length $2/5$ inch, which therefore must be the length of the equal intervals laid off on OX. Through each of the points of division of OX draw vertical lines, thus dividing the plane into a large number of small rectangles. Starting at A and sketching the diagonals of successive cornering rectangles, the locus ARS of $y = \cos^2 x$ is constructed.

From Fig. 118, it is seen that B always lies at the vertex of a right-angled triangle of hypotenuse OA. Thus as P describes the circle of radius OA, B describes a circle of radius $OA/2$. Therefore the curve $ARSX$ is related to the small circle ABO in the same manner that the curve of Fig. 59 is related to its circle; consequently the curve $ARSX$ of Fig. 118 is a sinusoid tangent to the x-axis. Thus the graph $y = \cos^2 x$ is a cosine curve of amplitude $1/2$ and wave length or period π, lying above the x-axis and tangent to it.

In Fig. 118, $OC = OH + OC = OH + HB \cos 2\theta = 1/2 + (1/2) \cos 2\theta$. Therefore the curve ARS has also the equation:

$$y = 1/2 + (1/2) \cos 2x \tag{1}$$

Hence we have a geometrical proof that

$$\cos^2 x = 1/2 + (1/2) \cos 2x \tag{2}$$

which is formula (5) of **§165.** Note that (1) is the curve $y = \cos 2x$ with its ordinates multiplied by $1/2$ then translated $1/2$ unit upward.

The curve $y = \sin^2 x$ is readily drawn in a manner similar to that above, by laying off the angle θ from OX as initial line. The curve is the same as that of Fig. 118, moved the distance $\frac{\pi}{4}$ to the right.

B. PLANE TRIANGLES: CONDITIONAL EQUATIONS

171. Law of Sines. The first of the conditional equations pertaining to the oblique triangle is a proportion connecting the sines of the three angles of the triangle with the lengths of the respective sides lying opposite. Call the angles of the triangle A, B, C, and indicate the opposite sides by the small letters a, b, c, respectively. From the vertex of any angle, drop a perpendicular p upon the opposite side, meeting the latter (produced if necessary) at D. Then, from the properties of right triangles, we have, from either Fig. 119 (1) or 119 (2)

$$p = c \sin DAB = a \sin C \tag{1}$$

But,

$$\begin{aligned} \sin DAB &= \sin A && \text{Fig. 119 (1)} \\ &= \sin (180° - A) && \text{Fig. 119 (2)} \\ &= \sin A \end{aligned}$$

Therefore: $$p = c \sin A = a \sin C \qquad (2)$$

Or: $$\mathbf{a / \sin A = c / \sin C} \qquad (3)$$

In like manner, by dropping a perpendicular from A upon a, we can prove:

$$\mathbf{b / \sin B = c / \sin C} \qquad (4)$$

Therefore: $$\mathbf{a / \sin A = b / \sin B = c / \sin C = 2R} \qquad (5)$$

Stated in words, the formula says: *In any oblique triangle the sides are proportional to the sines of the opposite angles.*

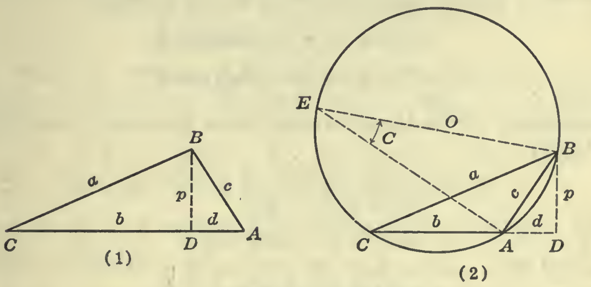


FIG. 119.—Derivation of the Law of Sines and the Law of Cosines.

GEOMETRICALLY: Calling each of the ratios in (5) $2R$, it is seen from Fig. 119 (2) that R is the radius of the circumscribed circle, and that $c/\sin C = 2R$ can be deduced from the triangle BAE, Similar construction can be made for the angles B and A.

172. Law of Cosines. From plane geometry we have the theorem: *The square of any side opposite an acute angle of an oblique triangle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides by the projection of the other side on it.* Thus in Fig. 119 (1):

$$a^2 = b^2 + c^2 - 2bd \qquad (1)$$

Now: $$d = c \cos A$$

Therefore: $$a^2 = b^2 + c^2 - 2bc \cos A \qquad (2)$$

Likewise we learn from geometry that the *square of any side opposite an obtuse angle of an oblique triangle is equal to the sum of the squares of the other two sides increased by twice the product of one of those sides by the projection of the other on it.* Thus in Fig. 119 (2):

$$a^2 = b^2 + c^2 + 2bd \tag{3}$$

Now: $d = c \cos DAB = c \cos (180 - A) = - c \cos A$
Therefore (3) becomes:

$$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A} \tag{4}$$

This is the same as (2), so that the trigonometric form of the geometrical theorem is the same whether the side first named is opposite an acute or opposite an obtuse angle.
In the same way we may show that, in any triangle:

$$\mathbf{b^2 = c^2 + a^2 - 2ca \cos B} \tag{5}$$

$$\mathbf{c^2 = a^2 + b^2 - 2ab \cos C} \tag{6}$$

Independently of the theorem from plane geometry, we note from Fig. 119 (1):

$$\begin{aligned} a^2 &= (b - d)^2 + p^2 = (b - d)^2 + c^2 - d^2 \\ &= b^2 + c^2 - 2bd \\ &= b^2 + c^2 - 2bc \cos A \end{aligned}$$

From 119 (2):

$$\begin{aligned} a^2 &= (b + d)^2 + p^2 = (b + d)^2 + c^2 - d^2 \\ &= b^2 + c^2 + 2bd \\ &= b^2 + c^2 + 2bc \cos DAB \\ &= b^2 + c^2 - 2bc \cos A \end{aligned}$$

since $DAB = 180° - A$ and $\cos (180° - A) = - \cos A$

Second Proof: Since any side of an oblique triangle is the sum of the projections of the other two sides upon it, the angles of projection being the angles of the triangle, we have:

$$\begin{aligned} \mathbf{a} &= \mathbf{b \cos C + c \cos B} \\ \mathbf{b} &= \mathbf{c \cos A + a \cos C} \\ \mathbf{c} &= \mathbf{a \cos B + b \cos A} \end{aligned} \tag{7}$$

Multiply the first of these equations by a, the second by b, the third by c, and subtract the second and third from the first. The result is;

$$\begin{aligned} a^2 - b^2 - c^2 &= ab \cos C + ca \cos B \\ &\quad - bc \cos A - ab \cos C \\ &\quad - ca \cos B - bc \cos A \\ &= -2bc \cos A \end{aligned}$$

or: $$a^2 = b^2 + c^2 - 2bc \cos A$$

173. Law of Tangents. An important relation results if we take formula (5) **§171** by composition and division. First write the law of sines in the form:

$$\frac{a}{b} = \frac{\sin A}{\sin B} \tag{1}$$

Then, by composition and division, the sum of the first antecedent and consequent is to their difference as the sum of the second antecedent and consequent is to their difference; that is:

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B} \tag{2}$$

Expressing the sums and difference on the right side of (2) by products by means of the formulas (1) and (2) of **§167,** we obtain:

$$\frac{a+b}{a-b} = \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)} \tag{3}$$

or simplifying and replacing the ratio of sine to cosine by the tangent, we obtain:

$$\mathbf{\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}} \tag{4}$$

In like manner it follows that:

$$\mathbf{\frac{b+c}{b-c} = \frac{\tan \frac{1}{2}(B+C)}{\tan \frac{1}{2}(B-C)}} \tag{5}$$

$$\mathbf{\frac{c+a}{c-a} = \frac{\tan \frac{1}{2}(C+A)}{\tan \frac{1}{2}(C-A)}} \tag{6}$$

Expressed in words: *In any triangle, the sum of two sides is to their difference, as the tangent of half the sum of the angles opposite is to the tangent of half of their difference.*

Geometrical Proof: From any vertex of the triangle as center, say C, draw a circle of radius equal to the shortest of the two sides of the triangle meeting at C, as in Fig. 120. Let the circle meet the side a at R and the same side produced at

E. Draw AE, AR. Call the angles at A, α, β, as shown. Then $BE = a + b$ and $BR = a - b$. Also:

$$\alpha + \beta = A$$

and: $\angle CRA = \beta + B$ (the external angle of a triangle RAB is equal to the sum of the two interior opposite angles), or $\alpha - \beta = B$.
Therefore:

$$\alpha = \tfrac{1}{2}(A + B)$$
$$\beta = \tfrac{1}{2}(A - B)$$

Draw $RS \parallel$ to EA. $\angle EAR = \angle ARS = 90°$
By similar triangles:

$$BE/BR = AE/SR$$
$$= \frac{AE}{AR} \div \frac{SR}{AR}$$

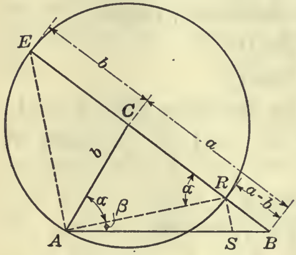


FIG. 120.—Geometrical Derivation of Law of Tangents.

But $BE = a + b$ and $BR = a - b$, while

$$\frac{AE}{AR} = \tan\alpha \text{ and } \frac{SR}{AR} = \tan\beta$$

Therefore: $\dfrac{a + b}{a - b} = \dfrac{\tan\ \frac{1}{2}(A + B)}{\tan\ \frac{1}{2}(A - B)}$

174. The following special formulas are readily deduced from the sine formulas and are sometimes useful as check formulas in computation. They are closely related to the law of tangents. From the proportion:

$$a:b:c = \sin A : \sin B : \sin C$$

by composition:

$$\frac{c}{a + b} = \frac{\sin C}{\sin A + \sin B}$$

Now by §**165** (1) and §**167** (1) this may be written:

$$\frac{c}{a + b} = \frac{2\sin\frac{1}{2}C\cos\frac{1}{2}C}{2\ \sin\ \frac{1}{2}(A + B)\ \cos\ \frac{1}{2}(A - B)}$$

Since $C = 180° - (A + B)$, therefore:

$$C/2 = 90° - \tfrac{1}{2}(A + B), \text{ and } \cos C/2 = \sin\tfrac{1}{2}(A + B)$$

Hence: $$\frac{\mathbf{c}}{\mathbf{a} + \mathbf{b}} = \frac{\sin\frac{1}{2}\mathbf{C}}{\cos\frac{1}{2}(\mathbf{A} - \mathbf{B})} = \frac{\cos\frac{1}{2}(\mathbf{A} + \mathbf{B})}{\cos\frac{1}{2}(\mathbf{A} - \mathbf{B})} \qquad (1)$$

In like manner it can be proved that:

$$\frac{\mathbf{c}}{\mathbf{a}-\mathbf{b}}=\frac{\sin\frac{1}{2}(\mathbf{A}+\mathbf{B})}{\sin\frac{1}{2}(\mathbf{A}-\mathbf{B})} \tag{2}$$

Both (1) and (2) can be readily deduced geometrically from Fig. 120.

175. The s-formulas. The cosine formula:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

can be written in the forms:

$$a^2 = (b + c)^2 - 2bc(1 + \cos A) \tag{1}$$

$$a^2 = (b - c)^2 + 2bc(1 - \cos A) \tag{2}$$

by adding $(+ 2bc)$ and $(- 2bc)$ to the right member in each case Now we know from **§166,** (1) and (3), that:

$$1 + \cos A = 2\cos^2(A/2)$$

$$1 - \cos A = 2\sin^2(A/2)$$

Therefore (1) and (2) above become:

$$a^2 = (b + c)^2 - 4bc\cos^2(A/2) \tag{3}$$

$$a^2 = (b - c)^2 + 4bc\sin^2(A/2) \tag{4}$$

writing these in the form:

$$4bc\sin^2(A/2) = a^2 - (b - c)^2 \tag{5}$$

$$4bc\cos^2(A/2) = (b + c)^2 - a^2 \tag{6}$$

and dividing the members of (5) by the members of (6), we obtain:

$$\tan^2(A/2) = \frac{a^2 - (b - c)^2}{(b + c)^2 - a^2} \tag{7}$$

Factoring the numerator and denominator we obtain:

$$\tan^2(A/2) = \frac{(a + b - c)(a - b + c)}{(b + c + a)(b + c - a)} \tag{8}$$

Let the perimeter of the triangle be represented by $2s$, that is, let:

$$a + b + c = 2s$$

Hence subtracting $2c$, $2b$, and $2a$ in turn:

$$a + b - c = 2s - 2c \text{ (subtracting } 2c)$$

$$a - b + c = 2s - 2b \text{ (subtracting } 2b)$$

$$b + c - a = 2s - 2a \text{ (subtracting } 2a)$$

Therefore equation (8) becomes:

$$\tan^2(A/2) = \frac{(s - b)(s - c)}{s(s - a)} \tag{9}$$

Let:

$$(s - a)(s - b)(s - c)/s = r^2 \tag{10}$$

then:

$$\tan^2 (A/2) = r^2/(s - a)^2$$

or:

$$\mathbf{\tan (A/2) = r/(s - a)} \tag{11}$$

Likewise:

$$\mathbf{\tan (B/2) = r/(s - b)} \tag{12}$$

$$\mathbf{\tan (C/2) = r/(s - c)} \tag{13}$$

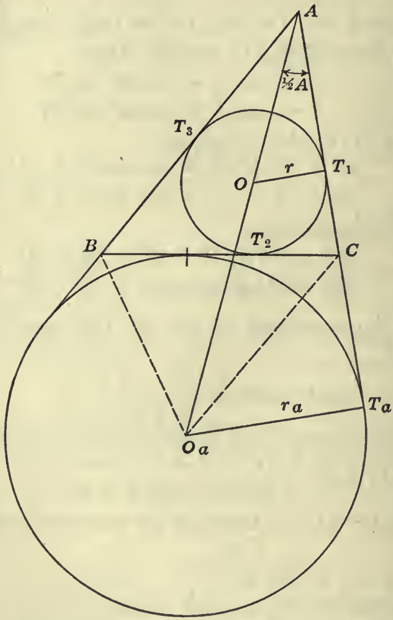


FIG. 121.—Geometrical Derivation of the s-Formulas.

GEOMETRICALLY: These formulas may be found by means of the diagram Fig. 121. Let the circle O be inscribed in the triangle ABC; its center is located at the intersection of the bisectors of the internal angles of the triangle. Let its radius be r.

Since $AT_1 = AT_3$, $BT_2 = BT_3$, $CT_1 = CT_2$, and since $2s = a + b + c$, it follows that one way of writing the value of s is:

$$s = BT_2 + T_2C + AT_1$$

Therefore:

$$AT_1 = s - a$$

Hence it follows that:

$$\tan (A/2) = r/(s - a) \tag{14}$$

Since this result is the same as (11) above, it proves that the r of equation (10) is the radius of the inscribed circle, and therefore proves that the radius of the inscribed circle may be expressed by the formula

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \tag{15}$$

a fact that is usually proved in text books on plane geometry.

177.* Miscellaneous Formulas for Oblique Triangles. The following formulas are given without proof. They are occasionally useful for reference, although no use will be made of them in this book. The following notation is used: The three sides of the oblique triangle are named a, b, c, and the angles opposite these A, B, C, respectively. The semi-perimeter of the triangle is s, or, $2s = a + b + c$. The radius of the circumscribed circle is R, that of the inscribed circle is r, and the radii of the escribed circles are r_a, r_b, r_c, tangent, respectively, to the sides a, b, c of the given triangle. K stands for the area of the triangle.

Then:

$$s = 4R \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C \tag{1}$$

$$s - c = 4R \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \cos \tfrac{1}{2}C \tag{2}$$

and analogs for $s - a$ and $s - b$.

$$r = 4R \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C \tag{3}$$

$$r_c = 4R \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \sin \tfrac{1}{2}C \tag{4}$$

and analogs for r_a and r_b.

$$r_a = s \tan \tfrac{1}{2}A,\ r_b = s \tan \tfrac{1}{2}B,\ r_c = s \tan \tfrac{1}{2}C \tag{5}$$

$$2K = ab \sin C = bc \sin A = ca \sin B \tag{6}$$

$$K = 2R^2 \sin A \sin B \sin C = \frac{abc}{4R} \tag{7}$$

$$K = \sqrt{s(s-a)(s-b)(s-c)} \tag{8}$$

$$K = rs = r_a(s-a) = r_b(s-b) = r_c(s-c) \tag{9}$$

$$K^2 = r r_a r_b r_c \tag{10}$$

$$K^2 = (s-a)\tan\tfrac{1}{2}A = (s-b)\tan\tfrac{1}{2}B = (s-c)\tan\tfrac{1}{2}C \tag{11}$$

C. NUMERICAL SOLUTION OF OBLIQUE TRIANGLES

178. An oblique triangle possesses six elements; namely, the three sides and the three angles. If any three of these six magnitudes be given (except the three angles), the triangle is determinate, or may be constructed by the methods explained in plane geometry; it will also be found that if any three of these six magnitudes be given, the other three may be computed by the formulas of trigonometry, provided, in both instances, that the given parts include at least one side.

It is convenient to divide the solution of triangles into four cases, as follows:

I. Given two angles and one side.
II. Given two sides and an angle opposite one of them.
III. Given two sides and the included angle.
IV. Given the three sides.

The solution of these cases with appropriate checks will now be given. The best arrangement of the work of computation usually consists in writing the data and computed results in the left margin of a sheet of ruled letter paper ($8\frac{1}{2}$ inches $\times$ 11 inches) and placing the computation in the body of the sheet. Every entry should be carefully labeled and computed results should be enclosed in square brackets. All work should be done on ruled paper and invariably in ink. Special calculation sheets (forms $M2$ and $M7$) have been prepared for the use of students. Neatness and systematic arrangement of the work and proper checking are more important than rapidity of calculation.

179. Computer's Rules. The following computer's rules are useful to remember in logarithmic work:

Last Digit Even: When it becomes necessary to discard a 5 that terminates any decimal, increase by unity the last digit retained if it be an odd digit, but leave it unchanged if it be an even digit; that is, *keep the last digit retained even.* Thus log π = 0.4971; hence write (1/2) log π = 0.2486. Also log sin 18° 5′ = 9.4900 + (correction) 19.5 = 9.4920.

Of course if the discarded figure is greater than 5, the last digit retained is increased by 1, while if the discarded figure is less than 5, the last digit retained is unchanged.

FUNCTIONS OF ANGLES IN SECOND QUADRANT: In finding from the table any function of an angle greater than 100° (but < 180°) *replace by their sum the first two figures of the number of degrees in the angle and take the cofunction of the result.* The method is valid because it is equivalent to the subtraction of 90° from the angle. By §**162** this always gives the correct numerical value of the function. The algebraic sign should be taken into account separately. Thus: sin 157° 32′ 7″ = cos 67° 32′ 7″. In case of an angle between 90° and 100°, ignore the first figure and proceed in the same way:

$$\tan 97^\circ\, 57'\, 42'' = -\cot 7^\circ\, 57'\, 42''$$

180. Case I. *Given two angles and one side,* as A, B, and c.

1. To find C, use the relation $A + B + C = 180°$.
2. To find a and b, use the law of sines, §**171.**
3. To check results, apply the check formula (1) or (2) §**172.**

EXAMPLE: In an oblique triangle, let $c = 1492$, $A = 49°\ 52'$, $B = 27°\ 15'$. It is required to compute C, a, b.

The following form of work is self explanatory. This arrangement, while readily intelligible to the beginner, does not conform to the proper standards of calculation explained above. It should be noted, however, that the process of work and the meaning of each number entering the calculation is properly indicated or labeled in the work.

To find C: $C = 180° - (A + B) = 103°\ 53'$

To find a:

As sin C	(103° 53′)	colog 0.0129
: c	(1492)	log 3.1738
: : sin A	(49° 52′)	log 9.8834
: a	[1175]	log 3.0701

To find b:

As	$\sin C$	$(103° 53')$	colog 0.0129
:	c	(1492)	log 3.1738
: :	$\sin B$	$(27° 15')$	log 9.6608
:	b	$[703.9]$	log 2.8475

Check:

As	$\sin \frac{1}{2}(A + B)$	$(38° 33.5')$	colog 0.2053
:	$\sin \frac{1}{2}(A - B)$	$(11° 18.5')$	log 9.2924
: :	c	(1492)	log 3.1738
:	$a - b$	$[469.4]$	log 2.6715

Also $a - b$ from first computations $= 471.1$ which checks 469.4, as computed, within 1.7.

The above work arranged in compact form appears as follows:

Computation of Triangle

c, A and B given

Data and results	To find a	To find b	Check
	$a = \dfrac{c \sin A}{\sin C}$	$b = \dfrac{c \sin B}{\sin C}$	$a - b = \dfrac{c \sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)}$
$c = 1492$	log 3.1738	log 3.1738	log 3.1738
$A = 49° 52'$	log sin 9.8034		
$B = 27° 15'$		log sin 9.6608	
$C = (103° 53')$	colog sin 0.0129	colog sin 0.0129	
$a = [1175]$	log 3.0701		
$b = [703.9]$		log 2.8475	
$(A - B)/2 = (11° 19.5')$			log sin 9.2924
$(A + B)/2 = (38° 33.5')$			colog sin 0.2053
$a - b = (471.1)$	Check!		[469.4] 2.6715

Examples

Find the remaining parts, given:

1. $A = 47° 20'$, $B = 32° 10'$, $a = 739$.

2. $B = 37° 38'$, $C = 77° 23'$, $b = 1224$.

3. $B = 25° 2'$, $C = 105° 17'$, $b = 0.3272$.

4. $C = 19° 35'$, $A = 79° 47'$, $c = 56.47$.

181. Case II. *Given two sides and an angle opposite one of them*, as a, b, and A.

1. To find B, use the law of sines, **§171**.
2. To find C, use the equation $A + B + C = 180°$.
3. To check, apply the check formula (1) or (2), **§174**.

When an angle as B, above, is determined from its sine, it admits of two values, which are supplementary to each other. There may be, therefore, two solutions to a triangle in Case II. The solutions are illustrated in Fig. 122.

In case one of the two values of B when added to the given angle A gives a sum greater than two right angles, this value of B must be discarded, and but one solution exists. If a be less than the perpendicular distance from C to c, no solution is possible.

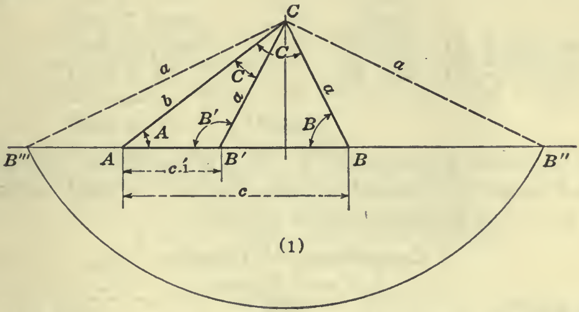


Fig. 122.—Case II of Triangles, for One, Two, and Impossible Solutions.

Example: Find all parts of the triangle if $a = 345$, $b = 534$, and $A = 25° 25'$.

The solution is readily understood from the following work.

To find B:	As a	(345)	colog	7.4622
	: b	(534)	log	2.7275
	: : $\sin A$	(25° 25′)	log	9.6326
	: $\sin B$	[41° 37′]	log	9.8223
	B'	[138° 23′]		

$C = 180° - (A + B) = 112° 58'$.

To find c:

As $\sin A$	(25° 25′)	colog	0.3674
: $\sin C$	(112° 58′)	log	9.9641
: : a	(345)	log	2.5378
: c	[740.1]	log	2.8693

Check:

As c	(740.1)	log	2.8693
: $b - a$	(189)	colog	7.7235
: : $\sin \frac{1}{2}(B + A)$	(33° 31′)	colog	0.2579
: $\sin \frac{1}{2}(B - A)$	[8° 6′]	log	9.1489
	Check!		9.9996

The sum of the logs should be 0. The discrepancy is 4 in the last decimal place.

To find c': $C' = 180° - A - B' = 16° 12'$

As $\sin A$	$(25° 25')$	colog	0.3674
$: \sin C'$	$(16° 12')$	log	9.4456
$:: a$	(345)	log	2.5378
$: c'$	$[224.3]$	log	2.3508

To Check:

As c'	(224.3)	log	2.3508
$: b - a$	(189)	colog	7.7235
$:: \sin \frac{1}{2}(B' + A)$	$(81° 54')$	colog	0.0043
$: \sin \frac{1}{2}(B' - A)$	$(56° 29')$	log	9.9210
	Check!		9.9996

The following arrangement of the work satisfies the requirements of properly arranged computation and is much to be preferred to the arrangement given above.

Computation of Triangle

a, b, and A given

Data and results	To find B	To find c	Check
	$\sin B = \dfrac{b \sin A}{a}$	$c = \dfrac{a \sin C}{\sin A}$	$\dfrac{c}{b-a} = \dfrac{\sin \frac{1}{2}(B + A)}{\sin \frac{1}{2}(B - A)}$
$a = 345$	colog 7.4622	log 2.5378	
$b = 534$	log 2.7275		
$A = 25° 25'$	log sin 9.6326	colog sin 0.3674	
$B = [41° 37']$	9.8223		
$C = (112° 58')$		log sin 9.9641	
$c = [740.1]$		log 2.8693	log 2.8693
$(B - A)/2 = (80° 6')$			log sin 9.1489
$(B + A)/2 = (33° 31')$			colog sin 0.2579
$b - a = (189)$			colog 7.7235
	Check!		9.9996
$B' = (138° 23')$			
$C' = (16° 12')$		log sin 9.4456	
$c' = [224.3]$		log 2.3508	log 2.3508
$(B' - A)/2 = (56° 29')$			log sin 9.9210
$(B' + A)/2 = (81° 54')$			colog sin 0.0043
$b - a = (189)$			colog 7.7235
	Check!		9.9996

Examples

Compute the unknown parts in each of the following triangles:

1. $a = 0.8$,	$b = 0.7$,	$B = 40° 15'$.
2. $a = 17.81$,	$b = 11.87$,	$A = 19° 9'$.
3. $b = 81.05$,	$c = 98.75$,	$C = 99° 19'$.
4. $c = 50.37$,	$a = 58.11$,	$C = 78° 13'$.
5. $a = 1213$,	$b = 1156$,	$B = 94° 15'$.

182. Case III. *Given two sides and the included angle,* as a, b, C.

1. To find $A + B$, use $A + B = 180° - C$.

2. To find A and B, compute $(A - B)/2$ by the law of tangents, **§173,** equation (4), then $A = (A + B)/2 + (A - B)/2$ and $B = (A + B)/2 - (A - B)/2$.

3. To find c, use law of sines, **§171.**

4. To check, use the check formula (2) **§174.**

EXAMPLE: Given $a = 1033$, $b = 635$, $C = 38° 36'$

$$A + B = 180° - 38° 36' = 141° 24'$$

To find A and B:

As $a + b$	(1668)	colog	6.7778
: $a - b$	(398)	log	2.5999
: : $\tan \frac{1}{2}(A + B)/$	(70° 42′)	log tan	0.4557
: $\tan \frac{1}{2}(A - B)/$	[34° 16′]	log tan	9.8334
	$A = 104° 58'$		
	$B = 36° 26'$		

To find c:

As $\sin A$	(104° 58′)	colog	0.0150
: $\sin C$	(38° 36′)	log	9.7951
: : a	(1033)	log	3.0141
: c	[667.1]	log	2.8242

Check:

As $\sin \frac{1}{2}(A - B)$	(34° 16′)	colog	0.2495
: $\sin \frac{1}{2}(A + B)$	(70° 42′)	log	9.9749
: : $a - b$	(398)	log	2.5999
: c	[667.2]	log	2.8243

Check!

An experienced computer would arrange the above work as follows:

Computation of Triangle

a, b, C given

Data and results	To find $A - B$	To find C	Check
	$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{a-b}{a+b}$	$c = \frac{a \sin c}{\sin A}$	$\frac{c}{a-b} = \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}$
$a = 1033$		log 3.0141	
$b = 635$			
$a - b = (398)$	log 2.5999		log 2.5999
$a + b = (1668)$	colog 6.7778		
$C = 38° 35'$		log sin 9.7951	
$(A + B)/2 = (70° 42')$	log tan 0.4557		log sin 9.9749
$(A - B)/2 = [34° 16']$	log tan 9.8334		colog sin 0.2495
$A = (104° 58')$		colog sin 0.0150	
$B = (36° 26')$			
$c = [667.1]$		log 2.8242	log 2.8243 Check!

Examples

Compute the unknown parts in each of the following triangles.

1.	$a = 78.9$,	$b = 68.7$,	$C = 78° 10'$.
2.	$c = 70.16$,	$a = 39.14$,	$B = 16° 16'$.
3.	$b = 1781$,	$c = 982.7$,	$A = 123° 16'$.
4.	$a = \pi$	$b = \pi/2$,	$C = \pi/3$.

183. Case IV. *Given the three sides.*

1. To find the angles, use the s-formulas, **§175**, (11), (12) and (13).

2. To check, use $A + B + C = 180°$.

EXAMPLE: Given $a = 455$, $b = 566$, $c = 677$, find A, B and C.

The following work is self explanatory. The work is arranged in final compact form, which, in this case, is as simple as any other possible arrangement.

Computation of Triangle

a, b, c given

Data and results	To find A, B, C
	$r^2 = (s-a)(s-b)(s-c)/s$ $\tan A/2 = r/(s-a)$. . .
$a = 455$	
$b = 566$	
$c = 677$	
$2s = 1698$	
$s = 849$	colog 7.0711
$s - a = 394$	log 2.5955
$s - b = 283$	log 2.4518
$s - c = 172$	log 2.2355
r^2	log 4.3539
r	log 2.1770
$A/2 = [20° 53']$	log tan 9.5815
$B/2 = [27° 58']$	log tan 9.7252
$C/2 = [41° 9']$	log tan 9.9415
$A = 41° 46'$	
$B = 55° 56'$	
$C = 82° 18'$	
Check! 180° 0'	

Exercises

Find the values of the angles in each of the following triangles:

1. $a = 173$, $b = 98.6$, $c = 230$.

2. $a = 8.067$, $b = 1.765$, $c = 6.490$.

3. $a = 1911$, $b = 1776$, $c = 1492$.

Miscellaneous Problems

The instructor will select only a limited number of the following problems for actual computation by the student. The student should be required, however, to *outline* in writing the solution of a number of problems which he is not required actually to compute, and, when practicable, to block out a suitable check for each one of them.

1. From one corner P of a triangular field PQR the side PQ bears N. 10° E. 100 rods. QR bears N. 63° E. and PR bears N. 38° 10′ E. Find the perimeter and area of the field.

2. The town B lies 15 miles east of A, C lies 10 miles south of A. X lies on the line BC, and the bearing of AX is S. 46° 20′ E. Find the distances from X to the other three towns.

3. To find the length of a lake (Fig. 123), the angle $C = 48° \, 10'$, the side $a = 4382$ feet, and the angle $B = 62° \, 20'$ were measured. Find the length of the lake c, and check.

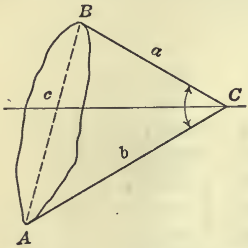


Fig. 123.—Diagram for Problem 3.

4. To continue a line past an obstacle L, Fig. 124, the line BC and the angles marked at B and C were measured and found to be 1842 feet, 28° 15′, and 67° 24′, respectively. Find the distance CD, and the angle at D necessary to continue the line AB; also compute the distance BD.

5. Find the longer diagonal of a parallelogram, two sides being 69.1 and 97.4 and the acute angle being 29° 34′.

What is the magnitude of the single force equivalent to two forces of 69.1 and 97.4 dynes respectively, making an angle of 29° 34′ with each other?

6. A force of 75.2 dynes acts at an angle of 35° with a force F. Their resultant is 125 dynes. What is the magnitude of F?

7. The equation of a circle is $\rho = 10 \cos \theta$. The points A and B on this circle have vectorial angles 31° and 54° respectively. Find the distance AB, (1) along the chord; (2) along the arc of the circle

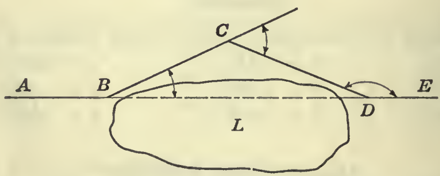


Fig. 124.—Diagram for Problem 4.

8. Find the lengths of the sides of the triangle enclosed by the straight lines:

$$\theta = 26°; \ \theta = 115°; \ \rho \cos(\theta - 45°) = 50.$$

9. A gravel heap has a rectangular base 100 feet long and 30 feet wide. The sides have a slope of 2 in 5. Find the number of cubic yards of gravel in the heap.

10. A point B is invisible and inaccessible from A and it is necessary to find its distance from A. To do this a straight line is run from A to P and continued to Q such that B is visible from P and Q. The following measurements are then taken: $AP = 2367$ feet; $PQ = 2159$ feet; $APB = 142° 37'.3$; $AQB = 76° 13'.8$. Find AB.

11. To determine the height of a mountain the angle of elevation of the top was taken at two stations on a level road and in a direct line with it, the one 5280 yards nearer the mountain than the other. The angles of elevation were found to be 2° 45′ at the further station and 3° 20′ at the nearer station. Find the horizontal distance ot the mountain top from the nearer station and the height of the mountain above it. Use S and T functions.

12. Explain how to find the distance between two mountain peaks M_1 and M_2, (1) when A and B at which measurements are taken are in the same vertical plane with M_1 and M_2; (2) when neither A nor B is in the same vertical plane with M_1 and M_2.

13. The sides of a triangular field are 534 yards, 679 yards and 474 yards. The first bears north, and following the sides in the order here given the field is always to the left. Find the bearing of the other two sides and the area.

14. From a triangular field whose sides are 124 rods, 96 rods, and 104 rods a strip containing 10 acres is sold. The strip is of uniform width, having as one of its parallel sides the longest side of the field. Find the width of the strip.

15. Three circles are externally mutually tangent. Their radii are 5, 6, 7 feet. Find the area and perimeter of the three-cornered area enclosed by the circles and the length of a wire that will enclose the group of three circles when stretched about them.

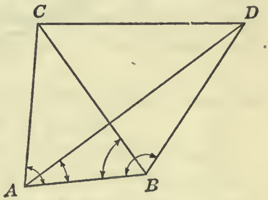


Fig. 125.—Diagram for Problem 16.

16. To find the distance between two inaccessible objects C and D, Fig. 125, two points A and B are selected from which both objects are visible. The distance AB is found to be 7572 feet. The following angles were then taken:

$$\begin{aligned} ABD &= 122° 37' \\ ABC &= 70° 12' \\ BAC &= 80° 20' \\ BAD &= 27° 13' \end{aligned}$$

Find the distance DC and check.

17. A circle of radius a has its center at the point (ρ_1, θ_1). Find its equation in polar coördinates. (Use law of cosines.)

18. A surveyor desired the distance of an inaccessible object O from A and B, but had no instruments to measure angles. He measured AA' in the line AO, BB' in the line BO; also AB, BA', AB'. How did he find OA and OB?

19. From a point A a distant object C bears N. 32° 16′ W. with angle of elevation 8° 24′; from B the same object bears N. 50° W. AB bears N. 10° 38′ W. The distance AB is 1000 yards. Find the distance AC.

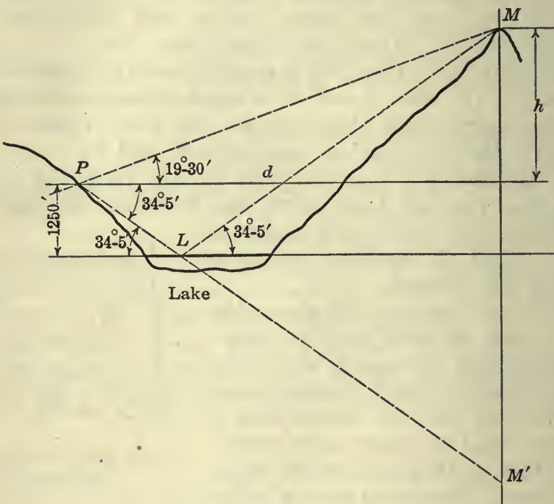


Fig. 126.—Diagram for Problem 20.

20. The angle of elevation of a mountain peak is observed to be 19° 30′. The angle of depression of its image reflected in a lake 1250 feet below the observer is found to be 34° 5′. Find the height of the mountain above the observer and the horizontal distance to it. (See Fig. 126.)

21. One side of a mountain is a smooth eastern slope inclined at an angle of 26° 10′ to the horizontal. At a station A a vertical shaft is sunk to a depth of 300 feet. From the foot of the shaft two horizontal tunnels are dug, one bearing N. 22° 30′ E. and the other S. 65° E. These tunnels emerge at B and at C respectively. Find the lengths of the tunnels and the lengths of the sides of the triangle ABC.

22. A rectangular field $ABCD$ has side $AB = 40$ rods; $AD = 80$ rods. Locate a point P in the diagonal AC so that the perimeter of the triangle APB will be 160 rods. (*Hint:* Express perimeter as a function of angle at P.)

23. Find the area enclosed by the lines $y = \frac{x}{2}$, $y = \sqrt{3}\, x$, and the circle $x^2 - 10x + y^2 = 0$. (*Hint:* Change to polar coördinates.)

24. The displacement of a particle from a fixed point is given by

$$d = 2.5 \cos t + 2.5 \sin t.$$

What values of t give maximum and minimum displacements; what is the maximum displacement?

25. A quarter section of land is enclosed by a fence. A farmer wishes to make use of this fence and 60 rods of additional fencing in making a triangular field in one corner of the original tract. Find the field of greatest possible area. Show that it is also the field of maximum perimeter, under the conditions given.

26. A force $F_1 = 100$ dynes makes an angle of $\theta°$ with the horizontal, and a second force $F_2 = 50$ dynes makes an angle of 90° with F_1. Determine θ so that (1) the sum of the horizontal components of F_1 and F_2 shall be a maximum; (2) so that the sum of the vertical components shall be zero.

27. Find the area of the largest triangular field that can be enclosed by 200 rods of fence, if one side is 70 rods in length.

28. Change the equation of the curve $xy = 1$ to polar coördinates, rotate through $-45°$ and change back to rectangular coördinates.

29. A particle moves along a straight line so that the distance varies directly as the sum $\sin t + \cos t$. When $t = \pi/4$, the distance is 10; find the equation of motion.

30. From the top of a lighthouse 60 feet high the angle of depression of a ship at anchor was observed to be 4° 52′, from the bottom of the lighthouse the angle was 4° 2′. Required the horizontal distance from the lighthouse to the ship and the height of the base of the lighthouse above the sea.

31. A vertical square shaft measuring 3 feet 6 inches on a side meets a horizontal rectangular tunnel 6 feet 6 inches high by 3 feet 6 inches wide. Find an expression for the length of a line AB shown in Fig. 127 when the angle θ is 37°.

32. University Hall casts a shadow 324 feet long on the hillside on which it stands. The slope of the hillside is 15 feet in 100 feet, and the elevation of the sun is 23° 27′. Find the height of the building.

33. To determine the distance of a fort A from a place B, a line BC and the angles ABC and BCA were measured and found to be 3225.5 yards, $60° 34'$, and $56° 10'$ respectively. Find the distance AB.

34. A balloon is directly over a straight level road, and between two points on the road from which it is observed. The points are 15,847 feet apart, and the angles of elevation are $49° 12'$ and $53° 29'$. Find the height.

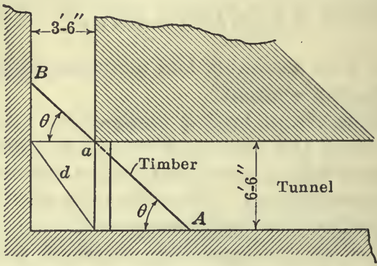


Fig. 127.—Diagram for Problem 31.

35. Two trees are on opposite sides of a pond. Denoting the trees by A and B, we measure $AC = 297.6$ feet, $BC = 864.4$ feet, and the angle $ABC = 87° 43'$. Find AB.

36. Two mountains are 9 and 13 miles respectively from a town, and they include at the town an angle of $71° 36'$. Find the distance between the mountains.

37. The sides of a triangular field are, in clockwise order, 534 feet, 679 feet, and 474 feet; the first bears north; find the bearings of the other sides and the area.

38. Under what visual angle is an object 7 feet long seen when the eye is 15 feet from one end and 18 feet from the other?

39. The shadow of a cloud at noon is cast on a spot 1600 feet west of an observer, and the cloud bears S., $76°$ W., elevation $23°$. Find its height.

CHAPTER X

WAVES

184. Simple Harmonic Motion. Let P be any point on a circle, and let D be the projection of P on any straight line in the plane of the circle. Then if the point P move uniformly (that is, so that equal distances are described in equal times) on the circle, the

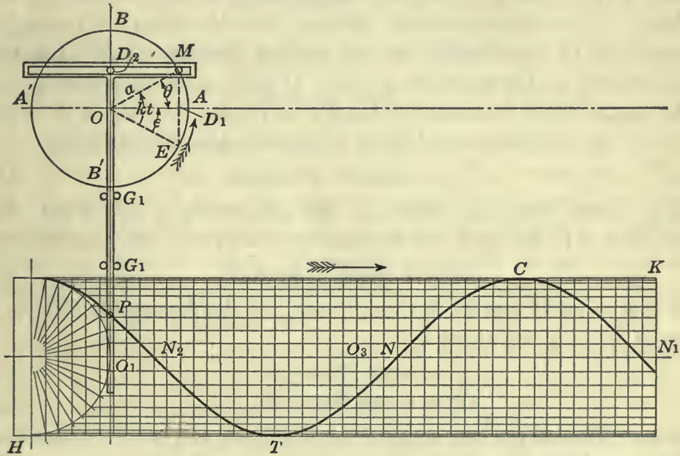


FIG. 128.—Mechanical Generation of Simple Harmonic Motion, and of a Simple Progressive Wave.

back-and-forth motion of the point D on the given straight line is called **simple harmonic motion.** On account of the frequency with which this term will occur, we shall abbreviate it by the symbols S.H.M. Fig. 128 illustrates a way in which this motion may be described by mechanical means. Let the uniformly rotating wheel OAB be provided with a pin M attached to its circumference, and free to move in the slot of the cross-head as shown, the arm of the cross-head being restricted to vertical motion by suitable guides G_1G_1. Then, as the wheel rotates, any point P of the arm of the cross-head describes simple harmonic motion in a vertical direction. The **amplitude** of the S.H.M. is the radius of the circle, or OB; its **period** is the time required for one complete revolution of the wheel.

In elementary physics it is explained that the motion of a simple pendulum is nearly simply harmonic. Also that the motion of a point of a vibrating violin string, or of a point of a tuning fork, is S.H.M. S.H.M. is a fundamental mode of motion of the particles of all elastic substances, and is therefore of great importance.

The motion of the point D_2 can readily be expressed by an equation, if the value of the angle θ be expressed in terms of the elapsed time t. Since the rotation is uniform, $\theta = kt$, where k is the angle described in one second, or the **angular velocity** of P. Let the radius OM of the circle be a feet. If O be taken as origin, and if the angle AOM be called θ, then if the point M was at A when $t = 0$, the displacement $OD_2 = y$ is given at any time t by

$$y = a \sin \theta = a \sin kt \tag{1}$$

In a similar way, the point D_1, the projection of M on OA, describes a S.H.M., and the displacement $OD_1 = x$ may be written

$$x = a \cos \theta = a \cos kt \tag{2}$$

If the point M was at E when $t = 0$, the displacements $OD_2 = y$ and $OD_1 = x$ are given by

$$\mathbf{y = a \sin (kt - \epsilon)} \tag{3}$$

$$\mathbf{x = a \cos (kt - \epsilon)} \tag{4}$$

when ϵ stands for the angle EOA; for $kt =$ angle EOM and $\theta = EOM - EOA$. In this equation $kt - \epsilon$ is called the **phase angle** and ϵ is called the **epoch angle** of the S.H.M.

These expressions may also be written in terms of the linear velocity V of M instead of the angular velocity k of OM. Let the uniform velocity of M be v feet per second. Since the radius OM is a feet, OM rotates at the rate of $v/a = k$ radians per second. This value of k may be substituted in equations (1) to (4).

It is obvious that

$$y = a \cos kt$$

represents a S.H.M. $\frac{\pi}{2}$ in advance of $y = a \sin kt$, since

$\sin\left(kt + \frac{\pi}{2}\right) = \cos kt$. A pair of S.H.M.'s possessing this property are said to be **in quadrature.** (1) and (2), or (3) and (4) may be said to be in quadrature.

The **period** of the S.H.M. $y = a \sin kt$ is the time T required for a complete revolution. If t_1 be the time at which M is at any given position, and if t_2 be the time at which M is next at the same position, then, since the angular velocity multiplied by the elapsed time gives the angular displacement, we have,

$$k(t_2 - t_1) = 2\pi$$

Therefore, since the difference $t_2 - t_1 = T$ is the period:

$$\mathbf{T = \frac{2\pi}{k}}$$

The number of complete periods per unit time is:

$$N = \frac{1}{T} = \frac{k}{2\pi}$$

N is called the **frequency** of the S.H.M.

It is obvious that all points of the moving cross-head, Fig. 128, describe S.H.M., and that (1) may be regarded as the equation of motion of any point of the cross-head if a suitable origin be selected. Thus (1) is the equation of motion of P referred to the origin O_1, where O_1 is the middle point of the up-and-down range of motion of P.

185. If P, Fig. 128, be a tracing point attached to the vertical arm of the cross-head and capable of describing a curve on a uniformly translated piece of smoked glass, HK, then when P describes S.H.M. in the vertical line OP, the curve N_1CTN_2P traced on the plate HK is a sinusoid, for the ordinates on HK measured with respect to the median line O_1N_1 are proportional to $\sin\theta$ and by hypothesis the abscissas or horizontal distances vary uniformly. If the plate HK move with exactly the same speed as the point M, the undistorted sinusoid of Fig. 59 is described, whose equation is

$$y = a \sin \frac{v}{a} t = a \sin \frac{x}{a} \qquad (1)[1]$$

[1] The student should note that $\frac{y}{a} = \sin \frac{x}{a}$ is of exactly the same shape as $y = \sin x$, for multiplying both ordinates and abscissas of any curve by a is merely constructing the curve to a different scale. However, $\frac{y}{3} = \sin \frac{x}{2}$ is a *distorted* sinusoid, for the ordinates of $y = \sin x$ are multiplied by 3 while the abscissas are multiplied only by 2.

where x is the abscissa of any point of the sinusoid referred to an origin (as N) moving with the plate. If, however, the velocity of the plate be v' instead of v, then the equation of the curve on HK, referred to axes moving with the plate, is of the form

$$y = a \sin \frac{v'}{a} t = a \sin \frac{x'}{a} = a \sin \frac{px}{a} = a \sin hx \qquad (2)$$

where $v't = x'$, $x' = px$, and $h = p/a$. Changing the relative speed of the wheel and plate corresponds to stretching or contracting the sine curve in the x direction.

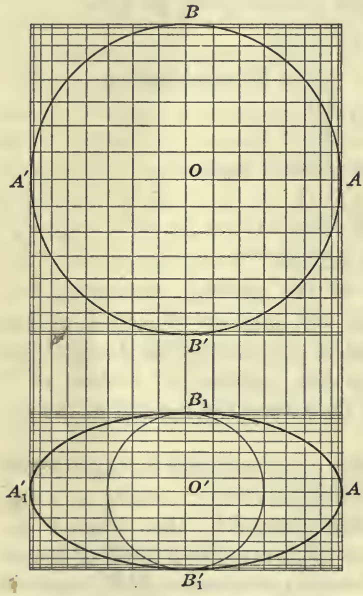


FIG. 129.—The Circle and the Ellipse Considered as Generated by Two S.H.M.'s in Quadrature.

186. Composition of Two S.H.M.'s at Right Angles. We have shown if a point M, moving uniformly on a circle, be projected upon both the X- and Y-axes, two S.H.M.'s result. The phase angles of these two motions differ from each other by $\frac{\pi}{2}$ or 90°. The converse of this fact, namely that uniform motion in a circle may be the resultant of two S.H.M. in quadrature, is easily proved, for the two equations of S.H.M.:

$$x = a \cos kt$$
$$y = a \sin kt$$

are obviously the parametric equations of a circle. Hence the theorem:

Uniform motion in a circle may be regarded as the resultant of two S.H.M.'s of equal amplitudes and equal periods and differing by $\pi/2$ in phase angle.

This important truth is illustrated by Fig. 129. Let the x- and y-axes be divided proportionally to the trigonometric sine, as in Fig. 59. Through the points of division of the two axes draw lines perpendicular to the axes, thus dividing the plane into a large number of small rectangles. Starting at the end of one of the axes, and sketching the diagonals of successive cornering rectangles, the circle $ABA'B'$ is drawn.

If the same construction be carried out for the case in which the y-axis is divided proportionally to $b \sin kt$ and in which the x-axis is divided proportionally to $a \sin kt$, the ellipse $A_1B_1A'_1B'_1$ results. These facts are merely a repetition of the statements made in §**74**.

Exercises

1. Draw a curve by starting at the intersection of *any two* lines of Fig. 129, and drawing the diagonals of successive cornering rectangles, and write the parametric equations of the curve.

2. Find the periods of the following S.H.M.:

$$
\begin{aligned}
&(a)\ y = 3 \sin 2t.\\
&(b)\ y = 10 \sin (1/2)t.\\
&(c)\ y = 7 \cos 4t.\\
&(d)\ y = a \sin 2\pi t.\\
&(e)\ y = a \sin (10t - \pi/3).\\
&(f)\ y = a \sin (2t/3 - 2\pi/5).\\
&(g)\ y = a \sin (bt + c).
\end{aligned}
$$

3. Give the amplitudes and epoch angles in each of the instances given in example 2.

4. The bob of a second's pendulum swings a maximum of 4 cm. each side of its lowest position. Considering the motion as rectilinear S.H.M. write its equation of motion.[1]

Write the equation of motion of a similar pendulum which was released from the end of its swing 1/2 second after the first pendulum was similarly released.

5. A particle moves in a straight line in such a way that its displacement from a fixed point of the line is given by $d = 2 \cos^2 t$. Show that the particle moves in S.H.M., and find the amplitude and period of the motion.

6. A particle moves in a vertical circle of radius 2 units with angular velocity of 20 radians per second. Counting time from the

[1] The term *period* is used differently in the case of a pendulum than in the case of S.H.M. The time of a *swing* is the period of a pendulum; the time of a *swing-swang* is the period of a S.H.M.

instant when the particle is at its lowest position, write the equation of motion of its projection (1) upon the vertical diameter; (2) upon the horizontal diameter; (3) upon the diameter bisecting the angle between the horizontal and vertical.

187. Waves. The curve described on the moving plate HK of Fig. 128, *if referred to coördinate axes moving with the plate,* is the *sinusoid* or *sine curve,* which for the sake of greater generality we shall suppose is of the type (§**185,** equation (2))

$$y = a \sin hx \tag{1}$$

If, however, we consider this curve as referred to the fixed origin O_1, then the *moving sinusoid* thus conceived is called a **simple progressive sinusoidal wave** or merely a **wave.** Under the conditions represented in Fig. 128, it is a wave progressing to the right with the uniform speed of the plate HK. At any *single instant,* the equation of the curve is:

$$y = a \sin h(x - O_1N) \tag{2}$$

where O_1N is the distance that the node N has been translated to the right of the origin O_1. If V be the uniform velocity of translation of HK, then:

$$O_1N = Vt \qquad (3)^1$$

and the equation of the wave is:

$$y = a \sin h(x - Vt)$$

or,

$$\mathbf{y = a \sin (hx - kt)} \tag{4}$$

if k be put for hV, so that

$$V = \frac{k}{h} \tag{5}$$

Because of the presence of the variable t, this is not the equation of a *fixed sinusoid,* but of a *moving sinusoid* or **wave.**

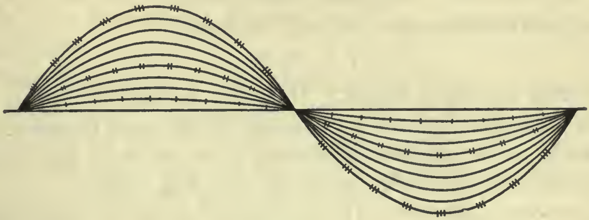
Applying the same terms used for S.H.M., the expression $(hx - kt)$ is the **phase angle,** the expression $(+kt)$ is the **epoch angle** and a is the **amplitude** of the wave. See Fig. 130a and c.

The expression $(hx - kt)$ is a *linear* function of the variables

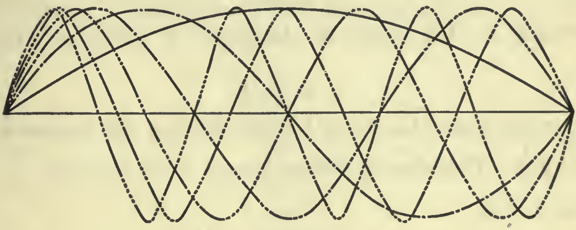
[1] In what follows, t is not the time elapsed since M, Fig. 128, was at A, as used in §**184,** but is the elapsed time since N was at O_1. These values of t differ by the time of half a revolution or by π/k.

x and t. The sine or cosine of this function is called a **simple harmonic function** of x and t.

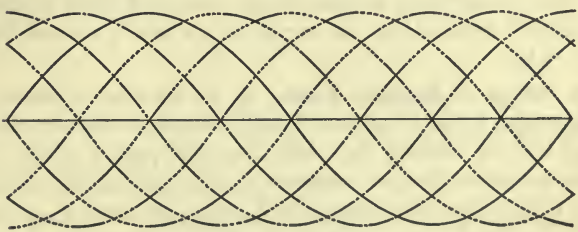
188. Wave Length. Since the period of the sine is 2π, if t remain constant and the expression hx be changed by the amount



a



b



c

FIG. 130.—Wave Forms, (*a*) of Different Amplitude; (*b*) of Different Wave Lengths; (*c*) of Different Phase or Epoch Angles.

2π, the curve (4) is translated to the left or right an amount such that trough coincides with trough and crest coincides with crest, and the curve in its second position coincides with the curve in its first position. Call x_2 the abscissa of any point of the curve in its second position whose original abscissa was x_1. Then:

$$hx_2 - hx_1 = 2\pi$$

or:

$$x_2 - x_1 = 2\pi/h$$

Calling the distance $x_2 - x_1 = L$, we have:

$$L = 2\pi/h \tag{1}$$

L is called the **wave length.** It is the distance from any crest to the next crest or from any trough to the next trough or from any node to the second succeeding node, or from any point of the wave to the next similar point. See Fig. 130*b*.

The wave length can also be determined in the following manner: The wave length of

$$y = \sin x \tag{2}$$

is obviously 2π, the length of the period of the sine. The sine curve

$$y = \sin hx \tag{3}$$

can be made from the above by multiplying the abscissas of all points by $\frac{1}{h}$. Therefore the wave length of the latter is $\frac{2\pi}{h}$. The wave length of

$$y = \sin (hx - kt) \tag{4}$$

must also be the same as that of (3), since the effect of the term kt is merely to translate the curve as a whole a certain distance to the right.

189. Period or Periodic Time. If we fix our attention upon any constant value of x, and if kt in (4) above be permitted to change by the amount 2π, then since the period of the sine is 2π, the curves at the two instances of time mentioned must coincide. Calling the two values of t, t_1 and t_2, we have by hypothesis

$$kt_2 - kt_1 = 2\pi$$

Writing:

$$t_2 - t_1 = T$$

we find

$$\mathbf{T = 2\pi/k} \tag{1}$$

The expression T is called the **periodic time or period** of the wave. It is the length of time required for the wave to move one wave length, or the length of time that elapses until trough again coincides with trough, etc. To contrast *wave length* and *period,* think of a person in a boat anchored at a fixed point in a lake. The time that the person must wait *at that fixed point* (x constant) for crest to follow crest is the *periodic time.* The *wave length* is the distance he observes between crests *at a given instant of time* (t constant).

190. Velocity or Rate of Propagation. The rate of movement V of the sinusoid on the plate HK, Fig. 128, is shown by equation (5), **§187,** to be k/h units of length per second. This is called the velocity of the wave or the velocity of propagation. The equation of the wave may be written:

$$y = a \sin h(x - Vt)$$

From equations (1) **§188** and (1) **§189** we may write

$$L = \frac{2\pi}{h}$$

$$T = \frac{2\pi}{h}$$

whence

$$\frac{k}{h} = \frac{L}{T}$$

Since $V = \frac{k}{h}$, we have:

$$\mathbf{V} = \frac{\mathbf{L}}{\mathbf{T}} \qquad (1)$$

This equation is obvious from general considerations, for the wave moves forward a wave length L in time T, hence the speed of the wave must be $\frac{L}{T}$.

191. Frequency. The number of periods per unit of time is called the **frequency** of the wave. Hence, if N represent the frequency of the wave,

$$\mathbf{N} = \frac{\mathbf{1}}{\mathbf{T}} = \frac{\mathbf{k}}{2\pi} \qquad (2)$$

There is no name given to the reciprocal of the wave length.

192. L and T Equation of a Wave. If we solve equations (1)

§188 and (1) **§189** for h and k respectively, and substitute these values of h and k in the equation

$$y = a \sin (hx - kt)$$

we obtain

$$\mathbf{y = a \sin 2\pi \left[\frac{x}{L} - \frac{t}{T}\right]} \qquad (1)$$

From this form it is seen that the argument of the sine increases by 2π when either x increases by an amount L or when t increases by the amount T. By use of (1), **§190,** the last equation may also be written:

$$\mathbf{y = a \sin \frac{2\pi}{L}(x - Vt)} \qquad (2)$$

193. Phase, Epoch, Lead. Consider the two waves:

$$y = a \sin \frac{2\pi}{L} (x - Vt) \qquad (1)$$

$$y = a \sin \frac{2\pi}{L}(x - Vt - E) \qquad (2)$$

The amplitudes, the wave lengths and the velocities are the same in each, but the second wave is in advance of the first by the amount E (measured in linear units), for the second equation can be obtained from the first by substituting $(x - E)$ for x, which translates the curve the amount E in the OX direction. In this case E is called the **lead** (or the **lag** if negative) of the second wave compared with the first.

The lead is a *linear* magnitude measured in centimeters, inches, feet, etc. The *epoch angle* is measured in radians. In the present case the epoch angle of (2) is $2\pi(Vt + E)/L$.

The terms **phase** and **epoch** are sometimes used to designate the *time*, or, more accurately, the *fractional amount of the period* required to describe the phase angle and epoch angle respectively. In this use, the *phase is the fractional part of the period that has elapsed since the moving point last passed through the middle point of its simple harmonic motion in the direction reckoned as positive.* See Fig. 130*c*.

The tidal wave in mid ocean, the ripples on a water surface, the wave sent along a rope that is rapidly shaken by the hand, are illustrations of progressive waves of the type discussed above.

Sound waves also belong to this class if the alternate condensations and rarefactions of the medium be graphically represented by ordinates. The ordinary progressive waves observed upon a lake or the sea are not, however, progressive waves of this type. The surface of the water in this case is not sinusoidal in form, but is represented by another class of curves known in mathematics as *trochoids*.

Exercises

1. Derive the amplitude, the wave length, the periodic time, the velocity of propagation of the following waves:

$$y = a \sin (2x - 3t).$$
$$y = 5 \sin (0.75x - 1000t).$$
$$y = 10 \sin \left(\frac{x}{2} - \frac{t}{3}\right)$$
$$y = 50 \sin \frac{2\pi}{7}(x - 3t).$$
$$y = 100 \sin \frac{2\pi}{25}(x - 20t - 4).$$
$$y = 100 \sin (5x + 4t).$$
$$y = 0.025 \sin \frac{2\pi}{4}(x + t/3).$$

2. Write the equation of a progressive sinusoidal wave whose height is 5 feet, length 40 feet and velocity 4 miles per hour.

3. Write the equation of a wave of wave length 10 meters, height 1 meter and velocity of propagation 3.5 miles per hour. (*Note:* 1 mile = 1.609 kilometers.)

4. Sound waves of all wave lengths travel in still air at 70° F. with a velocity of 1130 feet per second. Find the wave length of sound waves of frequencies 256, 128, 600 per second.

5. The lowest note recognizable as a musical tone was found by Helmholtz to possess about 40 vibrations per second. The highest note distinguishable by an ordinary ear possesses about 20,000 vibrations per second. If the velocity of sound in air be 1130 feet per second, find the wave length in each of these limiting cases, and write the equation of the waves if the amplitude be represented by the symbol a.

194. Stationary Waves. The form of a violin string during its free vibration is sinusoidal, but the nodes, crests, troughs, etc., are stationary and not progressive as in the case of the waves just discussed, and is therefore called a **stationary wave.** The water in a basin or even in a large pond or lake is also capable of vibrating in this way. Fig. 131 may be used to illustrate the stationary waves of this type, either of a musical string or of the water surface of a lake, but in the case of a vibrating string, the ends must be supposed to be fastened at the points O and N. The shores of the lake may be taken at I and K or at I and H, etc. As is well known, such bodies are capable of vibrating in segments so that the number of nodes may be large. This

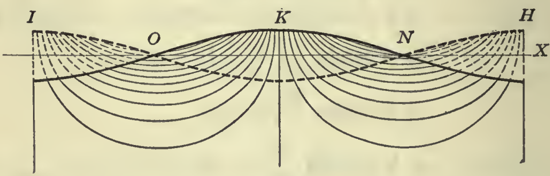


FIG. 131.—A Stationary Wave.

explains the "harmonics" of a vibrating violin string and the various modes in which *stationary waves* may exist on a water surface. A stationary wave on the surface of a lake or pond is known as a **seiche,** and was first noted and studied on Lake Geneva, Switzerland. The amplitudes of seiches are usually small, and must be studied by means of recording instruments so set up that the influence of progressive waves is eliminated. The maximum seiche recorded on Lake Geneva was about 6 feet, although the ordinary amplitude is only a few centimeters.

The equation of a stationary wave may be found by adding the ordinates of a progressive wave:

$$y = a \sin (hx - kt) \tag{1}$$

traveling to the right ($k > 0$), to the ordinates of a progressive wave:

$$y = a \sin (hx + kt) \tag{2}$$

traveling to the left.

Expanding the right members of (1) and (2) by the addition formula for the sine, and adding:

$$y = 2a \cos kt \sin hx \tag{3}$$

or in terms of L and T

$$y = 2a \cos \left(\frac{2\pi t}{T}\right) \sin \left(\frac{2\pi x}{L}\right) \tag{4}$$

In Fig. 131, the origin is at O and the X-axis is the line of nodes ONX. If we look upon $2a \cos kt$ as the variable amplitude of the sinusoid

$$y = \sin hx$$

we note that the nodes, etc., of the sinusoid remain stationary, but that the amplitude $2a \cos kt$ changes as time goes on. When $t = 0$, the sine curve has amplitude $2a$ and wave length $2\pi/h$. When $t = \pi/2k$ or $T/4$ the sinusoid is reduced to the straight line $y = 0$. When $t = \pi/k$ or $T/2$ the curve is the sinusoid:

$$y = -2a \sin hx$$

which has a trough where the initial form had a crest, and *vice versa*.

Exercises

In the following exercises the height of the wave means the maximum rise above the line of nodes. When a seiche is uninodal, the shores of the lake correspond to the points I and K, Fig. 131. When a seiche is binodal, the points I and H are at the lake shore.

1. From the equation of a stationary wave in the form $y = 2a \sin 2\pi x/L \cos 2\pi t/T$, show that K, Fig. 131, is at its lowest depth for $t = T/2, 3T/2, 5T/2, \ldots$

2. Henry observed a fifteen-hour uninodal seiche in Lake Erie, which was 396 kilometers in length. Write the equation of the principal or uninodal stationary wave if the amplitude of the seiche was 15 cm.

3. A small pond 111 meters in length was observed by Endros to have a uninodal seiche of period fourteen seconds. Write the equation of the stationary wave if the amplitude be a.

4. Forel reports that the uninodal longitudinal seiche of Lake Geneva has a period of seventy-three minutes and that the binodal seiche has a period of thirty-five and one-half minutes. The transverse seiche has a period of ten minutes for the uninodal and five minutes for the binodal. The longitudinal and transverse axes of the lake are

45 miles and 5 miles respectively. Write the equation of these different seiches.

5. A standing wave or uninodal seiche exists on Lake Mendota of period twenty-two minutes. If the maximum height is 8 inches and the distance across the lake is 6 miles, write the equation of the seiche.

195. Compound Harmonic Motion and Compound Waves. The addition of two or more simple harmonic functions of different periods gives rise to **compound harmonic motion.** Thus:

$$y = a \sin kt + b \sin 3\, kt$$

corresponds to the superposition of a S.H.M. of period $2\pi/3k$ and amplitude b upon a fundamental S.H.M. of period $2\pi/k$ and amplitude a. To compound motions of this type, there correspond compound waves of various sorts, such as a fundamental sound wave with overtones, or tidal waves in restricted bays or harbors. The graphs of the curves:

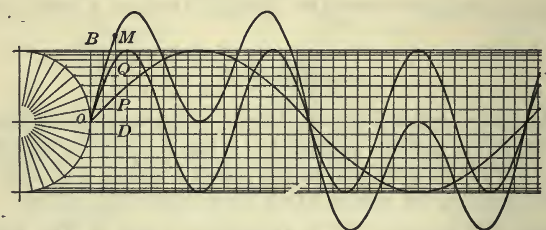


FIG. 132.—The Curves $y = \sin x$, $y = \sin 3x$ and the Compound Curve $y = \sin x + \sin 3x$.

$$y = \sin x + \sin 2x$$
$$y = \sin x + \sin 3x$$
$$y = \sin x + \sin 2x + \sin 3x$$

should be constructed by the student. They may be drawn by adding the ordinates of the various sinusoids constructed on the same axis, as in Fig. 132. To compound the curves, first draw the component curves, say $y = \sin x$ and $y = \sin 3x$ of Fig. 132. Then use the edge of a piece of paper divided proportionally to $\sin x$ (that is, like the scale OB, Fig. 132) and use this as a scale by means of which the successive ordinates of a given x

may be added. For example, to locate the point on the composite curve corresponding to the abscissa OD, Fig. 132, we must add DP and DQ. Hence place vertically at P the lower end of the paper scale just mentioned. The sixth scale division above P on this scale will then locate the required point M of the composite scale.

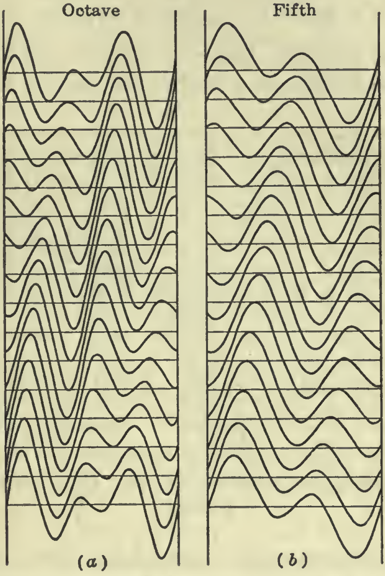


FIG. 133.—The Curves (a) $y = \sin x + \sin\ (2x + 2\pi n/16)$ and (b) $y = \sin\ 2x + \sin(3x + 2\pi n/16)$, for $n = 0, 1, 2, \ldots 15$. (*From Thomson and Tait.*)

In Fig. 133 the curves:

$$y = \sin x + \sin\ (2x + 2\pi n/16)$$
$$y = \sin 2x + \sin\ (3x + 2\pi n/16)$$

are shown for values of $n = 0, 1, 2, \ldots, 15$ in succession—that is, for successive phase differences corresponding to one-sixteenth of the wave length of the fundamental $y = \sin x$.

Wave forms compounded from the odd harmonics only are especially important, as alternating-current curves are of this type. See Fig. 134.

196. Harmonic Analysis. Fourier showed in 1822 in his "Analytical Theory of Heat" that a periodic single-valued function, say $y = f(x)$, under certain conditions of continuity, can be represented by the sum of a series of sines and cosines of the multiple angles of the form:

$$y = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ . \ . \ .$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ . \ . \ .$$

This means, for example, that it is always possible to represent the complex tidal wave in a harbor, by means of the sum of a

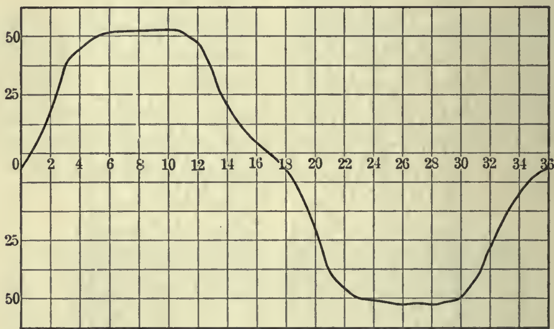


FIG. 134.—An Alternating Current Curve. Only Odd Harmonics are present.

number of simple waves or harmonics. The term **harmonic analysis** is given to the process of determining these sinusoidal components of a compound periodic curve. In **§195** we have performed the direct operation of finding the compound curve when the component harmonics are given. The inverse operation of finding the components when the compound curve is given is much more difficult, and its discussion must be postponed to a later course.

197.* Test for a Sinusoidal Function. Squared paper, known as semi-sinusoidal paper, has been prepared (see Fig. 135) with the horizontal scale divided proportionally to sin x and the vertical scale divided uniformly. The divisions are precisely the same as those in Fig. 59, except that the number of divisions is greatly increased. On this paper, the *sine curve is represented by a straight line drawn diagonally across the paper.* Since the sine curve appears as a straight line on this paper (just as the logarithmic curve appears as a straight line on semi-logarithmic paper) it is easy to test whether or not observed periodic data follow the law expressed by the sine curve. Thus the times of sunrise at Boston, Massachusetts, for the first day of each month have been plotted upon the sheet shown in Fig. 135. The points

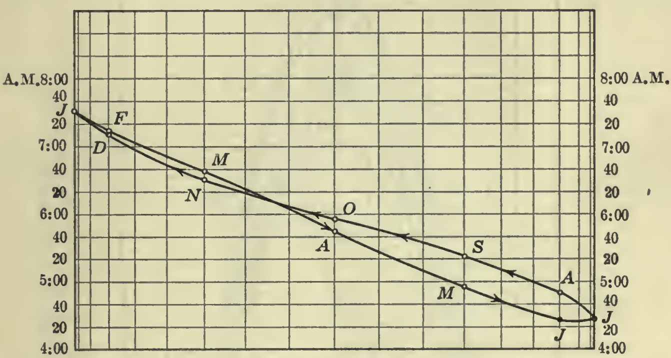


FIG. 135.—Time of Sunrise on the First Day of Each Month at Boston, Mass., Graphed Upon Semi-sinusoidal Paper.

corresponding to the various dates do not form a straight line, although it is obvious that the sine curve is a first approximation to the proper curve.

The times cf sunrise plotted in Fig. 135 are given in exercise (1) below.

Exercises

1. The times of sunset and sunrise at Boston, Massachusetts, for the first day of each month are as follows:

	J	F	M	A	M	J	J	A	S	O	N	D
Sunset	4:37	5:13	5:49	6:25	6:59	7:29	7:40	7:21	6:37	5:44	4:55	4:28
Sunrise	7:30	7:15	6:38	5:44	4:56	4:26	4:25	4:51	5:23	5:56	6:31	7:09
Length of day												

Graph the times of sunset upon semi-sinusoidal paper.

NOTE: The earliest sunset tabulated is December 1, which should be used for the date of the trough of the wave.

2. Determine the length of the day from the data of exercise 1, and graph the same upon semi-sinusoidal paper.

3. Determine whether the curves of Fig. 136 are sinusoidal or not.

198. Connecting Rod Motion. If one end of a straight line B be required to move on a circle while the other end of the line A moves on a straight line passing through the center of the circle,

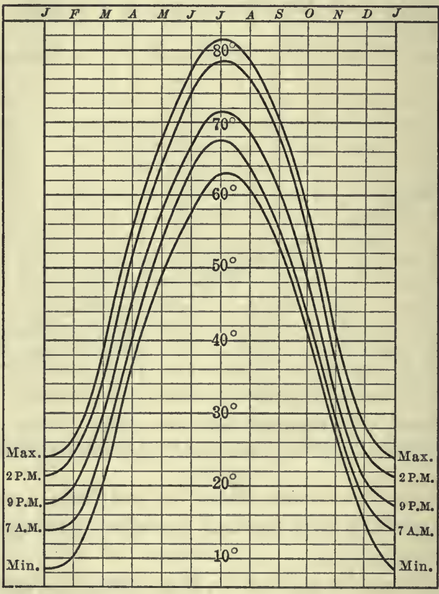


FIG. 136.—Daily Temperatures throughout the Year at Madison, Wis., Average of many years.

the resulting motion is known as **connecting rod motion.** The connecting rod of a steam engine has this motion, as the end attached to the crank travels in a circle while the end attached to the piston travels in a straight line. The motion of the end A of the connecting rod is approximately S.H.M. The approximation is very close if the connecting rod be very long in comparison with the diameter of the circle.

A second approximation to the motion of the point A can be shown to introduce the second harmonic or octave of the fundamental. In Fig. 137, let the radius of the circle be a and the length of the connecting rod be l. The length of the stroke MN is $2a$, and the origin may conveniently be taken at the mid point of the stroke, O. When B was at H, A was at M and when B was at K, A was at N. Then $MH = NK = l$, and $OC = l$. Now

$$x = CA - CO = CA - l = CD + DA - l \tag{1}$$

But

$$CD = a \cos \theta \tag{2}$$

and

$$\begin{aligned} DA &= \sqrt{l^2 - BD^2} \\ &= \sqrt{l^2 - a^2 \sin^2 \theta} \end{aligned} \tag{3}$$

Hence:

$$x = a \cos \theta + l \sqrt{1 - (a^2/l^2) \sin^2 \theta} - l \tag{4}$$

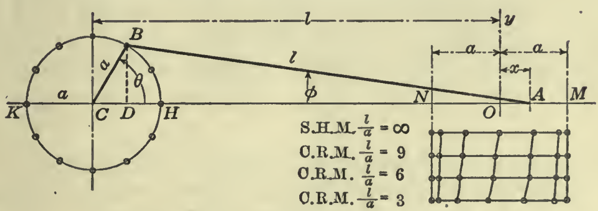


FIG. 137.—Connecting Rod Motion.

Approximating the radical by §111 ($\sqrt{1 - x} = 1 - x/2$) we obtain:

$$x = a \cos \theta + l\left(1 - \frac{a^2 \sin^2 \theta}{2l^2}\right) - l \tag{5}$$

But $\sin^2 \theta = (1 - \cos 2\theta)/2$, hence:

$$x = a \cos \theta + \frac{a^2}{4l} \cos 2\theta - \frac{a^2}{4l} \tag{6}$$

which is approximately true as long as l is much greater than a.

It is seen from the above result that the second approximation to connecting rod motion contains as overtone the *octave* or *second* harmonic, $\frac{a^2}{4l} \cos 2\theta$, in addition to the first or fundamental harmonic $a \cos \theta$.

Exercises

1. Draw the curve corresponding to equation (5) above if $a = 1.15$ inches, and $l = 3$ inches.

2. The motion of a slide valve is given by an equation of the form:

$$y = a_1 \sin(\theta + \epsilon) + a_2 \sin(2\theta + 90°).$$

Draw the curve if $a_1 = 100$, $a_2 = 25$, $\epsilon = 40°$.

3. Graph:

$$y = 5 \sin(\theta + 30°) + 2 \sin(2\theta + 90°).$$

4. Graph:

$$y = \sin x + (1/3) \sin 3x + (1/5) \sin 5x.$$

CHAPTER XI

COMPLEX NUMBERS

199. Scales of Numbers. To measure any magnitude, we apply a unit of measure and then express the result in terms of numbers. Thus, to measure the volume of the liquid in a cask we may draw off the liquid, a measure full at a time, in a gallon measure, and conclude, for example, that the number of gallons is $12\frac{1}{2}$. In this case the number $12\frac{1}{2}$ is taken from the *arithmetical scale* of numbers, 0, 1, 2, 3, 4, . . . If we desire to measure the height of a stake above the ground, we may apply a foot-rule and say, for example, that the height in inches above the ground is $12\frac{1}{2}$, or, if the positive sign indicates height *above* the ground, we may say that the height in inches is $+12\frac{1}{2}$. In this latter case the number $+12\frac{1}{2}$ has been selected from the *algebraic scale of* numbers . . . $-4, -3, -2, -1, 0, +1, +2, +3, +4,$ . . .

The scale of numbers which must be used to express the value of a magnitude depends entirely upon the nature of the magnitude. The attempt to express certain magnitudes by means of numbers taken from the algebraic scale may sometimes lead, as every student of algebra knows, to meaningless absurdities. Thus a problem involving the number of sheep in a pen, or the number of marbles in a box, or the number of gallons in a cask, cannot lead to a *negative* result, for the magnitudes just named are *arithmetical quantities* and their measurement leads to a number taken from the *arithmetical scale.* The absurdity that sometimes appears in results to problems concerning these magnitudes is due to the fact that one attempts to apply the notion of algebraic number to a magnitude that does not permit of it. Science deals with a great many different kinds of magnitudes, the measurement of some of which leads to arithmetical numbers while the measurement of others leads to algebraic numbers; the remarkable fact is that two different number scales serve adequately to express magnitudes of so many different sorts. The magnitudes of science are so various in kind that one might reasonably expect that the number of number systems required in the mathematics of these sciences would be very great.

The arithmetical scale, which includes integral and fractional numbers, is itself more general than is required for the expression of some magnitudes. For some magnitudes *fractions* are absurd—quite as absurd, in fact, as negative values are for other magnitudes. Thus the number of teeth on a gear wheel cannot be a *fraction.* The solution of the following problem illustrates this: "How many teeth must be cut on a pinion so that when driven by a spur gear wheel of fifty-two teeth it will revolve exactly five times as fast as the gear wheel?"

The arithmetical scale is used when we enumerate the number of gallons in a cask and say: 0, 1, 2, 3, . . . If we observe 3 gallons in the cask, and then remove one, we note those remaining and say *two;* we may remove another gallon and say *one;* we may remove the last gallon and say *zero;* but now the magnitude has come to an end—no more liquid may be removed.

Another conception of numerical magnitude is used when we measure in inches the height of a stake above the ground and say *three* We may drive the stake down an inch and say *two;* we may drive the stake another inch and say *one;* we may drive the stake another inch and say *zero*, or "level with the ground;" but, unlike the case of the gallons in the cask, we need not stop but may drive the stake another inch and say *one below the ground*, or, for brevity, *minus one;* and so on indefinitely, but always prefixing "minus" or "below the ground" or some expression that will show the relative position with respect to the zero of the scale. In this case we have made use of the algebraic scale of numbers.

Likewise, in estimating *time*, there is no zero in the sense of the gallons in the cask from which to reckon; we cannot conceive of an event so far past that no other event preceded it; we therefore select a standard event, and measure the time of other events with reference to the lapse *before* or *after* that; that is, we measure time by means of the *algebraic scale;* the symbols "B.C." or "A.D." could quite as well be replaced by the symbols "minus" and "plus" of the algebraic scale. The zero used is an arbitrary one and the magnitude exists in reference to it in two opposite senses, future and past, or, as is said in algebra, *positive* and *negative*. We are likewise obliged to recognize quantity as extending in two opposite senses from zero in the attempt to measure many other things; in locating points along an east and west line, no point is so far west that there are no other points west of it, hence the points could not be located on an arithmetical scale; the same in measuring force, which may be *attractive* or *repulsive;* or motion, which may be *toward* or *from*, or rotation, which may be *clockwise* or *anti-*

clockwise, etc. Because of the necessity of measuring such magnitudes, our notion of *algebraic number* has arisen.

Many of the magnitudes considered in science are completely expressed by means of arithmetical numbers only; for example, such magnitudes as *density* or specific gravity; *temperature;*[1] *electrical resistance; quantity of energy;* such as ergs, joules or foot-pounds; *power*, such as horse power, kilowatts, etc. All of the magnitudes just mentioned are **scalar**, as it is called; that is, they exist in one sense only—not in one sense and also in the opposite sense, as do forces, velocities, distances, as explained above. The arithmetical scale of numbers is therefore ample for their expression.

The distinction, then, between an algebraic number and an arithmetical number is the notion of *sense* which must always be associated with any algebraic number. Thus an algebraic number not only answers the question "*how many*" but also affirms the *sense* in which that number is to be understood; thus the algebraic number $+12\frac{1}{2}$, if arising in the measurement of angular magnitude, refers to an angular magnitude of $12\frac{1}{2}$ units (degrees, or radians, etc.) taken in the sense defined as *positive rotation.*

Exercises

Of the following magnitudes, state which may and which may not be represented adequately by an arithmetical number:

1. 10 volts.
2. 15 calories.
3. 25 dynes.
4. 2 kilograms.
5 20 miles per hour.
6. 4 acre-feet.
7. 180 revolutions per second.
8. 6-cylinder (engine).
9. 3 atmospheres.
10. 20 light-years.
11. 27° visual angle.
12. Atomic weight of oxygen.
13. 28 amperes.
14. $7\frac{1}{2}$ pounds per gallon.
15. 10° centigrade.
16. 272° absolute temperature.
17. 16 feet per second (velocity).
18. 32.2 feet per second per second (acceleration).
19. 200 gallons per minute.
20. 20 pounds per square inch.
21. 50 horse power.
22. 1.15 radians per second.
23. 30° latitude.
24. 14° angle of depression.
25. 18 cents per gallon.
26. 60 beats per minute.
27. 5360 feet above sea level.
28. 312 B.C.

[1] Temperature is an arithmetical quantity, since there is an *absolute zero* of temperature. Temperature does not exist in *two opposite senses*, but in *a single sense.*

200. Algebraic Number Not the Most General Sort. Algebraic numbers, although more general than arithmetical numbers, are themselves quite restricted. Sir William Hamilton, in order to emphasize the restricted character of an algebraic number, called algebra the "*science of pure time.*" That is, algebraic magnitude exists in the same restricted sense that time exists—because if we fix our attention upon any event, time exists in one sense (future) and in the exactly opposite sense (past), but *in no other sense at all.* Likewise, with the algebraic numbers, each number corresponds to a point of the algebraic scale (see §**1**); but for points not on the scale, or for points sidewise to the same, there corresponds no algebraic number. This is a way of saying that the algebraic scale is *one-dimensional;* Sir William Hamilton desired to emphasize this restriction by speaking of the "science of pure time," for it is of the very essence of the notion of time that it has one dimension and one dimension only. It is thus seen that there is an opportunity of enlarging our conception of number if we can remove the restriction of one dimension—that is, if we can get out of the line of the algebraic scale and set up a number system such that one number of the system will correspond, for example, to *each point of a plane,* and such that one point of the plane will correspond to each number of the system. We will seek therefore an extension or generalization of the number system of algebra that will enable us to consider, along with the points of the algebraic scale, those points which lie without it.

201. Numbers as Operators. The extension of the number system mentioned in the last section may be facilitated by changing the conception usually associated with symbols of number. The usual distinction in algebra is between *symbols of number* and *symbols of operation.* Thus a symbol which may be looked upon as answering the question "how many" is called a number, while a symbol which tells us to do something is called a **symbol of operation,** or, simply, an operator. Thus in the expression $\sqrt{2}$, $\sqrt{}$ is a symbol of operation and 2 is a number. A symbol of operation may always be read as a verb in the imperative mood; thus we may read $\sqrt{x}$: "Take the square root of x." Likewise $\log x$, and $\cos\theta$ may be read: "Find the logarithm of x," "Take the cosine of θ," etc. In these expressions "log" and "cos" are symbols of operation; they tell us to do something; they do not answer the question "how many" or "how much" and hence are not numbers. Here we speak of $\sqrt{\ }$, log, cos, as **operators**; we speak of x as the **operand**, or that which is operated upon.

It is interesting to note that *any number may be regarded as a symbol of operation;* by doing so we very greatly enlarge some original conceptions. Thus, 10 may be regarded not only as *ten*, answering the question "how many," but it may quite as well be regarded as denoting the *operation* of taking unity, or any other operand that follows it, ten times; to express this we may write $10 \cdot 1$, in which 10 may be called a **tensor** (that is, "stretcher"), or a symbol of the operation of stretching a unit until the result obtained is tenfold the size of the unit itself. In the same way the symbol 2 may be looked upon as denoting the operation of *doubling* unity, or the operand that follows it; likewise the tensor 3 may be looked upon as a *trebler*, 4 as a *quadrupler*, etc.

With the usual understanding that any symbol of operation operates upon that which follows it, we may write compound operators like $2 \cdot 2 \cdot 3 \cdot 1$. Here 3 denotes a trebler and $3 \cdot 1$ denotes that the unit is to be trebled, 2 denotes that this result is to be doubled and the next 2 denotes that *this* result is to be doubled. Thus representing the unit by a line running to the right, we have the following representation of the operators:

The unit $\rightarrow$
$3 \cdot 1 \quad \rightarrow\rightarrow\rightarrow$
$2 \cdot 3 \cdot 1 \quad \longrightarrow\longrightarrow$
$2 \cdot 2 \cdot 3 \cdot 1 \quad \longrightarrow\!\!\longrightarrow\longrightarrow\!\!\longrightarrow$

Notice the significance that should now be assigned to an exponent attached to these (or other) symbols of operation. The exponent means *to repeat the operation designated by the operator;* that is, the operation designated by the base is to be performed, and performed again on *this* result, and so on, the number of operations being denoted by the exponent. Thus 10^2 means to perform the operation of repeating unity ten times (indicated by 10) and then to perform the operation of repeating the result ten times, that is, it means $10\,(10 \cdot 1)$. Also, 10^3 means $10[10(10 \cdot 1)]$. Likewise $\log^2 30$ means $\log(\log 30)$ which, if the base be 10, equals log 1.4771, or finally 0.1694. An apparent exception occurs in the case of the trigonometric functions. The expression $\cos^2 x$ *should* mean, in this notation, $\cos(\cos x)$, but because trigonometry is historically so much older than the ideas here expressed, the expression $\cos^2 x$ came to be used as an abbreviation for $(\cos x)^2$, or $(\cos x) \times (\cos x)$.

To be consistent with the notation of elementary mathematics, the expression $\sqrt{4}$, looked upon as a symbol of operation, must denote an operation which must be performed *twice* in order to be equivalent to the operation of quadrupling; that is, such that $(\sqrt{4})^2 = 4$. Likewise $\sqrt[3]{4}$ denotes an operation which must be performed three times in succession in order to be equivalent to quadrupling. But we know that the operation denoted by 2, if performed twice, is equivalent to quadrupling; therefore $\sqrt{4} = 2$, etc. Just as 4^2, 4^3, etc., may be called *stronger* tensors than a single 4, so $\sqrt{4}$, $\sqrt[3]{4}$ may be called *weaker* tensors than the operator 4.

202. Reversor. The expression (-1), looked upon as a symbol of operation, is not a tensor, as it leaves the size unchanged of that upon which it operates. If this operator be applied to any magnitude, it will change the sense in which the magnitude is then taken to exactly the opposite sense. Thus, if 6 stands for six hours *after*, then $(-1)(6)$ stands for six hours *before* a certain event, and (-1) is the symbol of this operation of *reversing the sense of the magnitude.* Also if (6) stands for a line running six units to the *right* of a certain point, then $(-1)(6)$ stands for a line running six units to the *left* of that point, so that (-1) is the symbol which denotes the operation of turning the straight line through 180°. As 2, 3, 4, when looked upon as symbols of operations, were called tensors, the operator (-1) may conveniently be designated a **reversor.**

Exercises

Show graphically the effect of the operations indicated in each of the following exercises. Take as the initial unit-operand a straight line 1/2 inch long extending to the right of the zero or initial point. Explain each expression as consisting of the operand unity and symbols of operation—tensors, reversors, etc., which operate upon it one after the other in a definite order.

1. $2 \cdot 3 \cdot 1$.

2. $3 \cdot 3 \cdot 1$.

3. $-1 \cdot 3 \cdot 1$.

4. $2^3 \cdot 1$.

5. $\sqrt{3} \cdot 1$.

6. $(\sqrt[3]{2})^2 \cdot 1$.

7. $\sqrt{9} \cdot \sqrt{4} 1$.

8. $(\sqrt{4})^3 \cdot (-1) \cdot 1$.

9. $(-1)^3 \cdot 2^2 \cdot 3 \cdot 1$.

10. $3 \cdot 3^2 \cdot 1$.

11. $(-1)^2 \cdot 2^4 \cdot 1$.

12. $3 \cdot (-1) \cdot \sqrt{2} \cdot 1$.

13. $(\sqrt{2}) \cdot (-1)^{100} \cdot 1$.

14. $\sqrt[5]{10} \cdot 2 \cdot (-1) \cdot 1$.

15. A tensor, if permitted to operate seven times in succession, will just double the operand. Symbolize this tensor.

16. A tensor, if permitted to operate five times in succession, will quadruple the operand. Symbolize this tensor.

203. Versors. The expression $\sqrt{-1}$ cannot consistently, with the meaning already assigned to $\sqrt{\ }$ and (-1), be looked upon as answering the question "how many," and therefore is not a number in that sense; yet if we consider $\sqrt{-1}$ as a *symbol of operation*, it can be given a meaning consistent with the operators already considered. For if 2 is the operator that doubles, and $\sqrt{2}$ is the operator that *when used twice* doubles, then since (-1) is the operator that reverses, the expression $\sqrt{-1}$ should be an operator which, when used twice, reverses. So, as (-1) may be defined as the symbol which operates to turn a straight line through an angle of 180°, in a similar way *we may define the expression* $\sqrt{-1}$ *as a symbol which denotes the operation of turning a straight line through an angle of* 90° *in the positive direction.* The restriction of positive rotation is inserted in the definition merely for the sake of convenience.

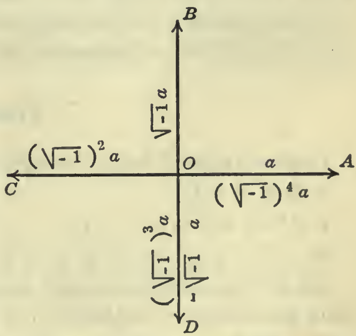


FIG. 138.—The Integral Powers of $\sqrt{-1}$.

The symbols (-1) and $\sqrt{-1}$ are not tensors. They do not represent a *stretching* or *contracting* of the operand. Their effect is merely to *turn* the operand to a new direction; hence these symbols may be called **versors,** or "turners."

204. Graphic Representation of $\sqrt{-1}$. In Fig. 138, let a be any line. Then a operated upon by $\sqrt{-1}$ (that is, $\sqrt{-1}\,a$) is turned *anti-clockwise* through 90°, which gives OB. Now, of course, $\sqrt{-1}$ can operate on $\sqrt{-1}\,a$ just as well as on a. Then $\sqrt{-1}\,(\sqrt{-1}\,a)$, or OC, is $\sqrt{-1}\,a$ turned positively through 90°. Similarly, $\sqrt{-1}\,[\sqrt{-1}(\sqrt{-1}\,a)]$ is $\sqrt{-1}\,(\sqrt{-1}\,a)$ turned through 90°, etc.

As we are at liberty to consider two turns of 90° as equivalent to one turn of 180°, therefore, $\sqrt{-1}(\sqrt{-1}\,a) = (-1)\,a$. Now $OD = (-1)\ OB$, $OD = (-1)\ (\sqrt{-1}\,a)$; but also $OD = \sqrt{-1}\ (-a)$, therefore, $(-1)\sqrt{-1}\,a = \sqrt{-1}\,(-a)$. Thus the student may show many like relations.

The operator $\sqrt{-1}$ is usually represented by the symbol i and will generally be so represented in what follows

Exercises

Interpret each of the following expressions as a symbol of operation:

1. $2,\ 3,\ 4,\ -1$.

2. $3^2,\ 2^3,\ 4^0,\ (-1)^2,\ (-1)^5$.

3. $\sqrt{2},\ \sqrt{3},\ \sqrt{-1},\ \sqrt[3]{2},\ \sqrt[3]{-1}$.

Select a convenient unit and construct each of the following expressions geometrically, explaining the meaning of each operator:

4. $2\cdot3\cdot5\cdot1$.

5. $2^3\cdot(-1)\cdot1$.

6. $3\cdot\sqrt{-1}\cdot2\cdot1$.

7. $(-1)^2\cdot\sqrt{-1}\cdot1$.

8. $2^2\cdot(-1)^3\cdot(\sqrt{-1})^0\cdot1$.

9. $3\cdot\sqrt{-1}\cdot(-1)\cdot\sqrt{-1}\cdot1$.

205. Complex Numbers. The explanation of the meaning of the symbol $(a+bi)$ will be given in the following section. It will be shown in subsequent theorems that any expression made up of the sum, product, power or quotient of real numbers and imaginaries may be put in the form $a+bi$, in which both a and b are real. The expression $a+bi$ is therefore said to be the **typical form** of the imaginary. An expression of the form $a+bi$

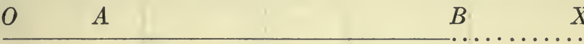
is also called a **complex number,** since it contains a term taken from each of the following scales, so that the unit is not single but double or complex:

$$\cdots -3,\ -2,\ -1, 0, +1, +2,\ +3,\ \ldots$$
$$\cdots -3i, -2i, -i,\ 0, +i,\ +2i, +3i,\ \ldots$$

It is important to note that the only element common to the two series in this complex scale is 0.

206. Meaning of a Complex Number. Any real number, or any expression containing only real numbers, may be considered as locating a point in a *line.*

Thus, suppose we wish to draw the expression $2 + 5$. Let O be the zero point and OX the positive direction. Lay off $OA = 2$ in the direction OX and at A lay off $AB = 5$ in the direction OX. Then the path $OA + AB$ is the geometrical representation of $2 + 5$.



Any complex number may be taken as the representation of the position of a point in a plane. For, suppose $c + di$ is the complex number. Let O be the zero point and OX the positive direction. Lay off $OA = +c$ in the direction OX and at A erect di in the direction OY, instead of in the direction OX as in the last example. See Fig. 139. It is agreed to consider the step to the right, OA, followed by the step upward, AP, as the *meaning* of the complex number $c + di$. *Either the broken path $OA + AP$ or the direct path OP may be taken as the representation of $c + di$, and either path constitutes* **the definition of the sum of c and di.**

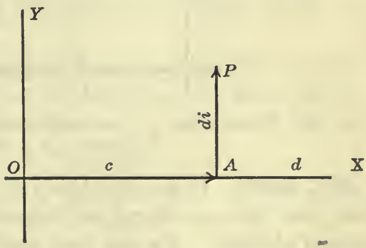


FIG. 139. The Geometrical Construction of a Complex Number, $c + di$.

In tbe same manner $c - di$, $-c - di$ and $-c + di$ may be constructed.

The meaning of some of the laws of algebra as applied to imaginaries may now be illustrated. Let us construct $c + di + a + bi$.

The first two terms, $c + di$, give $OA + AB$, locating B (Fig. 140). The next two terms, $a + bi$, give $BC + CP$, locating P. Hence the entire expression locates the point P with reference to O. Now if the original expression be changed in any manner allowed by the laws of algebra, the result is merely a different path to the same point. Thus:

$c + a + di + bi$ is the path OA, AD, DC, CP
$(c + a) + (d + b)i$ is the path OD, DP
$a + di + c + bi$ is the path OE, EH, HC, CP
$a + di + bi + c$ is the path OE, EH, HF, FP, etc.

The student should consider other cases. Are there any method of locating P with the same four elements, which the figure does not illustrate?

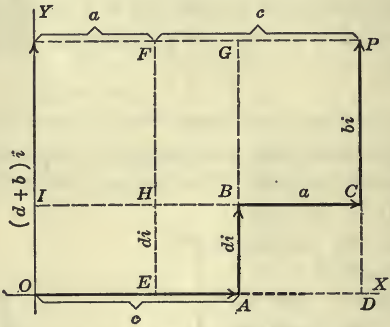


FIG. 140. Illustration of the Application of the Laws of Algebra to the Expression $c + di + a + bi$.

207. Laws. It can be shown by simple geometrical construction that the operator i, as defined above, obeys the ordinary laws of algebra. We can then apply all of the elementary laws of alegbra to the symbol i and work with it just as we do with any other letter. The following are illustrations of each law:

COMMUTATIVE LAW:

$$c + di + a + bi = c + a + di + bi = di + c + bi + a, \text{ etc.}$$
$$ai = ia, \qquad iai = iia = aii, \text{ etc.}$$

The equation $10\sqrt{-1} = \sqrt{-1}\cdot 10$, or better, $10\sqrt{-1}\cdot 1 = \sqrt{-1}\cdot 10\cdot 1$ may be said to mean that the result of performing the operation of turning unity through 90° and performing upon the result the operation of taking it ten times, is the same as the result of performing the operation of taking unity ten times and performing upon this result the operation of turning through 90°.

ASSOCIATIVE LAW:

$$(c + di) + (a + bi) = c + (di + a) + bi, \text{ etc.}$$
$$(ab)i = a(bi) = abi, \text{ etc.}$$

DISTRIBUTIVE LAW:

$$(a + b)i = ai + bi, \text{ etc.}$$

The expression $\sqrt{-a}$, where a is any number of the arithmetical scale, is defined as equivalent to $\sqrt{-1 \cdot a}$; that is, $\sqrt{-a} = i\sqrt{a}$. For example, $\sqrt{-4} = 2i$, $\sqrt{-3} = i\sqrt{3}$, etc. *In what follows it is presupposed that the student will reduce expressions of the form $\sqrt{-a}$ to the form $i\sqrt{a}$ before performing algebraic operations.* From this it follows that $\sqrt{-a}\sqrt{-b} = -\sqrt{ab}$ *and not* $\sqrt{ab}$.

The relation $\sqrt{-4} = 2\sqrt{-1}$ may be interpreted as follows: (-4) is the operator that quadruples and reverses; then $\sqrt{-4}$ is an operator which used twice quadruples and reverses. But $2\sqrt{-1}$ is an operator such that two such operators quadruple and reverse. That is, $\sqrt{-4} = 2\sqrt{-1}$.

208. Powers of i. We shall now interpret the powers of i by means of the new significance of an exponent and by the commutative, associative and other laws. First:

i^0 or $i^0\,1$	$= +1$	$i^5 = i^4 i$	$= i$
i^1 or $i^1\,1$	$= i$	$i^6 = i^5 i$	$= -1$
i^2	$= -1$	$i^7 = i^6 i$	$= -i$
$i^3 = i^2 i$	$= -i$	$i^8 = i^7 i$	$= +1$
$i^4 = i^2 i^2$	$= +1$	etc.	etc.

Whence it is seen that all even powers of i are either $+1$ or -1, and all odd powers are either i or $-i$. The student may reconcile this with Fig. 138. The zero power of i must be unity, for the exponent zero can only mean that the operation denoted by the symbol of operation *is not to be performed at all;* that is, unity is to be left unchanged; thus 10^0 or $10^0 \cdot 1 = 1$, $2^0 = 1$, $\log^0 x = x$, $\sin^0 x = x$, etc.

Exercises

Select as unit a distance 1/2 inch in length extending to the right and represent graphically each of the following expressions:

1. $i + 2i^2 + 3i^3 + 4i^4 + \ . \ . \ .$

2. $i + i^4 + i^5 + i^8 + i^9 + \ . \ . \ .$

3. $i + i^4 + i^7 + i^8 + i^9 + i^{12} + \ . \ . \ .$

4. $i(i + i^4 + i^7 + i^8 + i^9 + i^{12} + \ . \ . \ . \ ).$

5. $i + i^2 + i^3 + 2i^4 + i^5 + i^6 + i^7 + 3i^8 + \ . \ . \ .$

209. Two complex numbers are said to be conjugate if they differ only in the sign of the term containing $\sqrt{-1}$. Such are $x + iy$ and $x - iy$.

Conjugate imaginaries have a real sum and a real product.

For: $(x + yi) + (x - yi)$
$= x + yi + x - yi$, by associative law
$= x + x + yi - yi$, by commutative law
$= 2x + (yi - yi)$, by associative law
$= 2x + (y - y)i$, by distributive law

Likewise: $(x + yi)(x - yi)$
$= x(x - yi) + yi(x - yi)$, by distributive law
$= x^2 - xyi + yix - yiyi$, by distributive law
$= x^2 - y^2i^2 + xyi - xyi$, by commutative law
$= x^2 + y^2 + (xy - xy)i$, by distributive law and by substituting $i^2 = -1$
$= x^2 + y^2$

It is well to note that the *product of two conjugate complex numbers is always positive and is the sum of two squares.*

This fact is very important and will be frequently used. Thus $(3 - 4i)(3 + 4i) = 3^2 + 4^2 = 25$; $(1 + i)(1 - i) = 2$
$(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta = 1$, etc.

210. *The sum, product, or quotient of two complex numbers is, in general, a complex number of the typical form $a + bi$.*

Let the two complex numbers be $x + yi$ and $u + vi$.

(1) Their sum is $(x + yi) + (u + vi)$
$= x + yi + u + vi$
$= x + u + yi + vi$
$= (x + u) + (y + v)i$

by the laws of algebra. This last expression is in the form $a + bi$.

(2) Their product is $(x + yi)(u + vi)$

$$\begin{aligned} &= x(u + vi) + yi(u + vi) \\ &= xu + xvi + yiu + yivi \\ &= xu + yvi^2 + xvi + yui \\ &= (xu - yv) + (xv + yu)i \end{aligned}$$

by the laws of algebra. This last expression is in the form $a + bi$.

(3) Their quotient is

$$\frac{x + yi}{u + vi} = \frac{(x + yi)(u - vi)}{(u + vi)(u - vi)}$$

By the preceding, the numerator is of the form $a' + b'i$. By **§209,** the denominator equals $u^2 + v^2$. Then the quotient equals

$$\frac{a' + b'i}{u^2 + v^2} = \frac{a'}{u^2 + v^2} + \frac{b'}{u^2 + v^2}i$$

by distributive law. This last expression is of the form $a + bi$.

Exercises

Reduce the following expressions to the typical form $a + bi$; the student must change every imaginary of the form $\sqrt{-a}$ to the form $i\sqrt{a}$.

1. $\sqrt{-25} + \sqrt{-49} + \sqrt{-121} - \sqrt{-64} - 6i$.

2. $(2\sqrt{-3} + 3\sqrt{-2})(4\sqrt{-3} - 5\sqrt{-2})$.

3. $(x - [2 + 3i])(x - [2 - 3i])$.

4. $(-5 + 12\sqrt{-1})^2$.

5. $(3 - 4\sqrt{-1})^2$.

6. $(\sqrt{1 + i})(\sqrt{1 - i})$.

7. $(\sqrt{e} - \sqrt{-e})^2$.

8. $\dfrac{a}{\sqrt{-1}}$.

9. $\dfrac{2}{3 + \sqrt{-2}}$.

10. $\dfrac{56}{1 - \sqrt{-7}}$.

11. $\dfrac{1 + i}{1 - i^2}$.

12. $\dfrac{1}{(1 - i)^6}$.

13. $\dfrac{1 - i^3}{(1 - i)^3}$.

14. $\dfrac{1 - 2\sqrt{-3}}{1 + 2\sqrt{-3}}$.

15. $\dfrac{(2 + 3\sqrt{-1})^2}{2 + \sqrt{-1}}$.

16. $\dfrac{a + xi}{a - xi} - \dfrac{a - xi}{a + xi}$.

23

211. Irrational Numbers. A **rational number** is a number that can be expressed as the quotient of two integers. All other real numbers are **irrational.** Thus $\sqrt{2}$, $\sqrt{5}$, $\sqrt{7}, \pi$, e, are irrational numbers. An irrational number is always intermediate in value to two rational numbers which differ from each other by a number as small as we please. Thus

$$1.414 < \sqrt{2} < 1.415$$
$$1.4142 < \sqrt{2} < 1.4143$$
$$1.41421 < \sqrt{2} < 1.41422, \text{ etc.}$$

It is easy to prove that $\sqrt{2}$ cannot be expressed as the quotient of two integers. For, if possible, let

$$\sqrt{2} = \frac{a}{b} \tag{1}$$

where a and b are integers and $\frac{a}{b}$ is in its lowest terms. Squaring the members of (1) we have

$$2 = \frac{a^2}{b^2} \tag{2}$$

This cannot be true, since 2 is an integer and a and b are prime to each other.

An irrational number, when expressed in the decimal scale, is never a repeating decimal. For if the irrational number could be expressed in that manner, the repeating decimal could be evaluated by §**123** in the fractional form $\frac{a}{1 - r}$, which, by definition of an irrational number, is impossible. On the contrary, *every rational number when expressed in the decimal scale is a repeating decimal.* Thus $1/3 = 0.33\ldots$ and $1/4 = 0.25000\ldots$.

The proof that π and e are irrational numbers is not given in this book.

See *Monographs on Modern Mathematics*, edited by J. W. A. Young.

The student should not get the idea that because irrational numbers are usually approximated by decimal fractions, that the irrational number itself is not exact. This can be illustrated by the graphical construction of $\sqrt{2}$. Locate the point P whose coördinates are (1,1). Call the abscissa OD and the ordinate DP. Then $OP = \sqrt{2}$ and $OD = 1$, $DP = 1$. It is obvious that the hypotenuse OP must be considered just as *exact* or definite as the legs OD and DP. The notion that irrational numbers are inexact must be avoided.

The process of counting objects can be carried out by use of the primitive scale of numbers 0, 1, 2, 3, 4, . . . The other numbers made use of in mathematics, namely,

(1) positive and negative numbers
(2) integral and fractional numbers
(3) rational and irrational numbers
(4) real and imaginary numbers

may be looked upon as classes of numbers that permit the operations *subtraction, division* and *evolution,* to be carried out under all circumstances. Thus, in the history of algebra it was found that in order to carry out subtraction under all circumstances, negative numbers were required; to carry out division under all circumstances, fractions were required; to carry out evolution of arithmetical numbers under all circumstances, irrational numbers were required; finally to carry out evolution of algebraic numbers under all circumstances, imaginaries were required. It will be found that it will not be necessary to introduce any additional form of number into algebra; that is, the most general number required is a number of the form $a + bi$, where a and b are positive or negative, integral or fractional, rational or irrational. This is the most general number that satisfies the following conditions:

(a) The possibility of performing the operations of algebra and the inverse operations under all circumstances.

(b) The conservation or permanence of the fundamental laws of algebra: namely, the commutative, associative, distributive and index laws.

Further extension of the number system beyond that of complex numbers leads to operators which do not obey the commutative law in multiplication; that is, in which the value of a product is dependent upon the order of the factors, and in which a product does not necessarily vanish when one factor is zero. Numbers of this kind the student may later study in the introduction to the study of electromagnetic theory under the head of "Vector Analysis" or in the subject of "Quaternions." Such numbers or operators do not belong to the domain of numbers we are now studying.

212. *If a complex number is equal to zero, the imaginary and real parts are separately equal to zero.*

Suppose $$x + y\sqrt{-1} = 0$$

then $$x = -y\sqrt{-1}$$

Now it is absurd or impossible that a real number should equal an imaginary, except they each be zero, since the real and imaginary scales are at right angles to each other and intersect only at the point zero.

Therefore: $$x = 0 \text{ and } y = 0$$

If two complex numbers are equal, then the real and the imaginary parts must be respectively equal.

For if $$x + yi = u + vi$$

then $$(x - u) + (y - v)i = 0$$

Whence, by the above theorem,

$$x - u = 0 \text{ and } y - v = 0$$

That is, $$x = u \text{ and } y = v$$

213. Modulus. Let the complex number $x + yi$ be constructed, as in Fig. 141, in which $OA = x$ and $AP = yi$. Draw the line OP, and let the angle AOP be called θ.

The numerical length of OP is called the **modulus** of the complex number $x + yi$. It is algebraically represented by $\sqrt{x^2 + y^2}$, or by $|x + yi|$. Thus, mod $(3 + 4i) = \sqrt{9 + 16} = 5$.

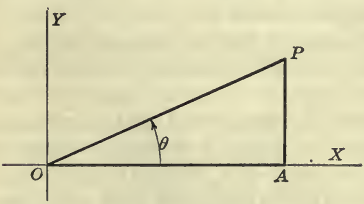


FIG. 141.—Modulus and Amplitude of a Complex Number.

The student can easily see that *two conjugate complex numbers have the same modulus, which is the positive value of the square root of their product.*

If $y = 0$, the mod $(x + yi) = \sqrt{x^2} = |x|$, where the vertical lines indicate that merely the numerical or absolute value of x is called for. Thus the modulus of any real number is the same as what is called the *numerical* or *absolute* value of the number. Thus, mod $(-5) = 5$.

214. Amplitude. In Fig. 141 the angle AOP or θ is called the **argument** or **amplitude** or simply the **angle** of the complex number $x + yi$.

Putting $r = \sqrt{x^2 + y^2} = \text{mod } (x + yi)$, we have

$$\sin \theta = \frac{y}{r} \qquad \cos \theta = \frac{x}{r}$$

Therefore,

$$x + yi = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta)$$

in which we have expressed the complex number $x + yi$ in terms of its modulus and amplitude.

To put $3 - 4i$ in this form, we have:

$$\text{mod } (3 - 4i) = \sqrt{9 + 16} = 5; \quad \sin \theta \doteq \frac{y}{r} = -\frac{4}{5}; \quad \cos \theta = \frac{x}{r} = \frac{3}{5}$$

Therefore,

$$(3 - 4i) = 5\left[\frac{3}{5} - \frac{4}{5}i\right]$$

The amplitude θ is $\tan^{-1}\left(-\frac{4}{5}\right)$, and is in the *fourth* quadrant. Why?

It is well to plot the complex number in order to be sure of the amplitude θ. It avoids confusion to use positive angles in all cases. For example, to change $3 - \sqrt{3}\,i$ to the polar form, plot the point $(3, -\sqrt{3})$ and find from the triangle that $r = 2\sqrt{3}$ and $\theta = 330°$. Hence

$$3 - \sqrt{3}\,i = 2\sqrt{3}(\cos 330° + i \sin 330°)$$

The amplitude of all positive numbers is 0, and of all negative numbers is 180°. The unit expressed in terms of its modulus and amplitude is evidently $1(\cos 0 + i \sin 0)$.

215. Vector. The point P, located by $OA + AP$ or $x + yi$, may also be considered as located by the line or radius vector OP; that is, by a line starting at O, of length r and making an angle θ with the direction OX. A directed line, as we are now considering OP, is called a **vector.** When thus considered, the two parts of the compound operator

$$r\,(\cos \theta + i \sin \theta) \tag{1}$$

receive the following interpretation: The operator $(\cos \theta + i \sin \theta)$, which depends upon θ alone, turns the unit lying along OX through an angle θ, and may therefore be looked upon as a *versor* of rotative power θ. The versor $(\cos \theta + i \sin \theta)$ is often abbreviated by the convenient symbol **cis** θ. The operator r is a tensor, which stretches the turned unit in the ratio $r : 1$. The result of these two operations is that the point P is located r units from O in a direction making the angle θ with OX.

The operator (1) above is also represented by the notation (r, θ), for example $(5, \angle 30°)$. Expression (1) is called the **polar form** of the complex number $(x + iy)$.

Thus, the operator $(\cos\theta + i\sin\theta)$ is simply a more general operator than i, but of the same kind. The operator i turns a unit through a right angle and the operator $(\cos\theta + i\sin\theta)$ turns a unit through an angle θ. If θ be put equal to 90°, $\cos\theta + i\sin\theta$ reduces to i.

For: $\theta = 0$, $\cos\theta + i\sin\theta$ reduces to 1
$\theta = 90°$, $\cos\theta + i\sin\theta$ reduces to i
$\theta = 180°$, $\cos\theta + i\sin\theta$ reduces to -1
$\theta = 270°$, $\cos\theta + i\sin\theta$ reduces to $-i$

Since $3 - 4i = 5(\frac{3}{5} - \frac{4}{5}i)$ the point located by $3 - 4i$ may be reached by turning the unit vector through an angle $\theta = \sin^{-1}(-4/5) = \cos^{-1} 3/5$ and stretching the result in the ratio $5:1$.

If a complex number vanishes, its modulus vanishes; and conversely, if the modulus vanishes, the complex number vanishes.

If $x + yi = 0$, then $x = 0$ and $y = 0$, by §**212**. Therefore, $\sqrt{x^2 + y^2} = 0$. Also, if $\sqrt{x^2 + y^2} = 0$, then $x^2 + y^2 = 0$, and since x and y are real, neither x^2 nor y^2 is negative, and so their sum is not zero unless each be zero.

If two complex numbers are equal, their moduli are equal, but if two moduli are equal, the complex numbers are not necessarily equal.

If $x + yi = u + vi$, then $x = u$ and $y = v$ by §**212**.

Therefore, $\sqrt{x^2 + y^2} = \sqrt{u^2 + v^2}$

But if $\sqrt{x^2 + y^2} = \sqrt{u^2 + v^2}$, obviously x^2 need not equal u^2 nor $y^2 = v^2$.

216. Sum of Complex Numbers. Let a given complex number locate the point A, Fig. 142, and let a second complex number locate the point B. Then if the first of the complex numbers be represented by the radius vector OA and if the second complex number be represented by the radius vector OB, the sum of the two complex numbers will be represented by the diagonal OC of the parallelogram constructed on the lines OA and OB. This law of addition is the well-known *law of addition of vectors* used in physics when the resultant of two forces or the resultant of two velocities, two accelerations, or two directed magnitudes of any kind, is to be found.

The proof that the sum of the two complex numbers is represented by the diagonal OC is very simple. Let the graph of the first complex number be $OD_1 + D_1A$ and let that of the second be $OD_2 + D_2B$. To add these, at the point A construct $AE = OD_2$ and $EC = D_2B$. Then the sum of the two complex numbers is geometrically represented by $OD_1 + D_1A + AE + EC$, or by the radius vector OC which joins the end points. Since, by construction, the triangle AEC is equal to the triangle OD_2B, therefore AC

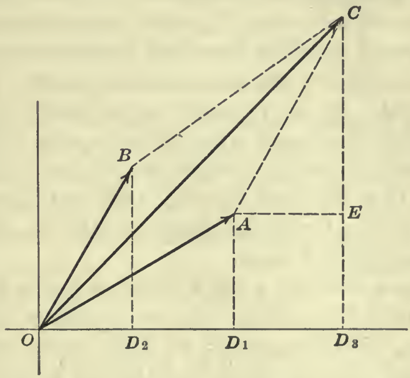


FIG. 142.—Sum of Two Complex Numbers.

must be equal and parallel to OB, so that the figure $OACB$ is a parallelogram, and OC, which represents the required sum, is the diagonal of this parallelogram, which we were required to prove.

Exercises

Find algebraically the sum of the following complex numbers, and construct the same by means of the law of addition of vectors.

1. $(1 + 2i) + (3 + 4i)$.

2. $(1 + i) + (2 + i)$.

3. $(1 - i) + (1 + 2i)$.

4. $(3 - 4i) - (3 + 4i)$.

5. $(-2 + i) + (0 - 4i)$.

6. $(-1 + i) + (3 + i) + (2 + 2i)$.

7. $(2 - i) + (-2 + i) + (1 + i)$.

8. Find the modulus and amplitude (in degrees and minutes) of $2(\cos 30° + i \sin 30°) + (\cos 45° + i \sin 45°)$.

9. By the parallelogram of vectors, show that the sum of two conjugate complex numbers is real.

10. If R be the sum of the complex numbers $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z_3 = x_3 + y_3 i$, etc., show that $-R, z_1, z_2, z_3, \ldots$ form the sides of a closed polygon.

217. Polar Diagrams of Periodic Functions. Three methods of representing simple periodic phenomena have already been explained: (1) *Crank* or *clock* diagrams as shown in Fig. 128 and explained in §**184**; (2) the *sine curve* or sinusoid in rectangular coördinates, as shown in Fig. 59 and explained in §**55**; (3) *polar diagrams*, in which the circle (twice drawn) corresponds to a crest and trough of the sine curve, as shown in Fig. 63 and explained in §**64**. As the principal application of these methods is to phenomena that vary with the time, one of the variables may conveniently be taken to represent time or a constant multiple of time. Thus the angle θ in the crank and polar diagrams or the abscissa in the Cartesian diagram, may be represented by a constant multiple of t as ωt.

The difference between a **clock diagram** and a **polar diagram** of a simple periodic function may be stated as follows: In a clock diagram, a rotating vector of fixed length OP is continuously projected upon a fixed line OX; in a polar diagram, a stationary line of fixed length OA is continuously projected upon a rotating radius vector OP. See Figs. 52 and 63.

Each of the three methods possesses a peculiar advantage of its own, but probably the best insight with regard to periodic phenomena is given by the polar diagram. In each, the complete period of the phenomena is represented by one complete revolution of the radius vector. The polar method is not only well adapted to represent simply varying periodic phenomena, in which case the polar diagram is a circle passing through the origin, but it is equally well adapted to represent cases in which the periodic motion is compounded from a number of simple harmonics. In many important cases in science, especially in the phenomena of alternating electric currents, only the *odd* harmonics are commonly present as components of the resulting motion. The equation of compound harmonic motion in rectangular coördinates, in which only odd harmonics appear, is of the form:

$$y = a_1 \sin \omega t + a_3 \sin 3\omega t + a_5 \sin 5\omega t + \ . \ . \ . \qquad (1)$$

in which a_1, a_3 . . . are any constants and in which ωt has replaced x. If the epochs[1] of the various harmonics be t_1, t_3, t_5, . . . the proper form of the equation would be:

$$y = a_1 \sin \omega(t - t_1) + a_3 \sin 3\omega(t - t_3) + a_5 \sin 5\omega(t - t_5) + \ . \ . \ . \qquad (2)$$

A curve of type (1) or (2) must represent a pattern within the interval $\omega t = \pi$ to $\omega t = 2\pi$ which is the opposite of the pattern presented within the interval $\omega t = 0$ to $\omega t = \pi$; for increasing ωt by the amount π in each of the components:

$$\sin \omega t, \sin 3\omega t, \sin 5\omega t \ . \ . \ .$$

of the compound motion has the effect of changing the algebraic sign of each term, but leaves the absolute value unchanged. This is because the sine curve, and all of the odd harmonics of the sine curve, are just reversed in sign by adding a straight angle (180°) or an odd number of straight angles to the original angle. Hence y has the same sequence of values, but of opposite signs, within each of the two half-intervals of the period 2π. Fig. 143 illustrates this. The curve A is the graph of an alternating current wave (after Fleming) in rectangular coördinates, while the same function is shown in polar coördinates by curve B. It is observed that the second portion of the Cartesian curve is exactly similar to the first portion, except that its position with reference to the x-axis is reversed. In the polar diagram this truth is brought out by the fact that the loop that represents the "trough" of the wave is identical with the loop that represents the "crest" of the wave, that is, the curve is twice drawn to represent the interval of a complete period from $\omega t = 0$ to $\omega t = 2\pi$.

If only even harmonics are present, the equation of the curve in rectangular coördinates is of the form:

$$y = a_0 + a_2 \sin 2\omega t + a_4 \sin 4\omega t + \ . \ . \ . \qquad (3)$$

or, if the epochs are not zero,

$$y = a_0 + a_2 \sin 2\omega(t - t_2) + a_4 \sin 4\omega(t - t_4) + \ . \ . \ . \qquad (4)$$

[1] This expression instead of "epoch angle" is the proper term in this case as t is measured in units of time and not in angular measure. The epoch angles are ωt_1, ωt_2, etc.

Because of the factor 2 in each harmonic term, the period of the function may be considered π instead of 2π, so that the sequence of values of y are repeated in each interval 0 to π, π to 2π, etc., and not reversed in sign as in the case of the odd harmonics.

The fact that the pattern for each interval π is repeated right side up and not reversed is illustrated by the graph of

$$y = 1 + \sin 2\omega t + \sin 4\omega t \qquad (5)$$

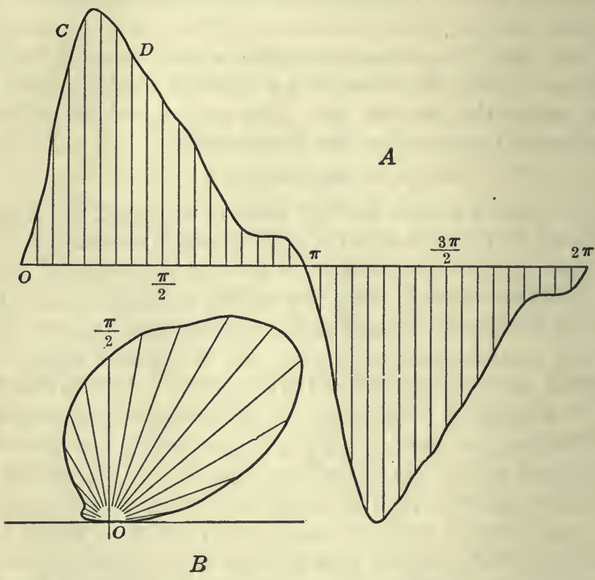


FIG. 143.—Rectangular and Polar Alternating Current Curves. (*After Fleming.*)

shown in Fig. 144. The effect of the constant term is, of course, merely to raise the graph a distance of one unit.

In *form*, curves with only even harmonics present do not differ from curves with both odd and even, for substituting $t' = 2t$, the general case (equation (3)) becomes:

$$y = a_0 + a_2 \sin \omega t' + a_4 \sin 2\omega t' + a_6 \sin 3\omega t' + \ . \ . \ . \qquad (6)$$

which contains both odd and even harmonics in t', and is of period 2π. The curve (3) is of the same shape as (6) but of period π.

A curve made up of both odd and even harmonics may have any form consistent with its one-valued and continuous character. The portion of the curve above the x-axis (if any) need not have the same form as the part below; the only essential is that the curve for

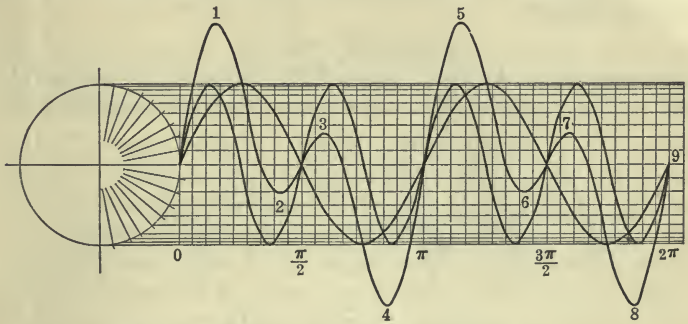


Fig. 144.—Graph of $y = \sin 2\,\omega t + \sin 4\,\omega t$.

each successive interval of 2π be a repetition of the curve in the preceding interval.

In polar coördinates, a curve made up of only even harmonics is *described but once* as θ varies from 0° to 360°. In general such

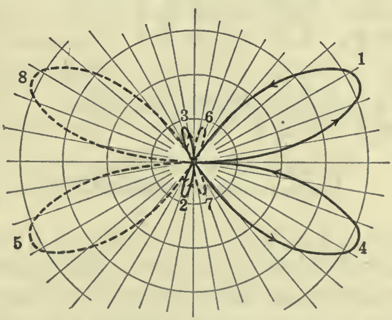


Fig. 145.—Graph of $\rho = \sin 2\theta + \sin 4\theta$.

curves have more "loops" than curves made up of odd harmonics, for the loops of the odd harmonics are *twice drawn* as θ varies from 0° to 360°. Thus the curves:

$$\rho = \sin\,\theta,\ \rho = \sin 3\theta,\ \rho = \sin 5\theta,\ \ldots$$

have 1, 3, 5, . . . loops respectively *each twice drawn.* The curves

$$\rho = \sin 2\theta,\ \rho = \sin 4\theta,\ \rho = \sin 6\theta,\ \ldots$$

have 4, 8, 12, . . . loops respectively, *each once drawn.* Also the curve:

$$\rho = \sin 2\theta + \sin 4\theta \tag{7}$$

is represented by the heavy curve of Fig. 145 as θ varies from 0° to 180° and by the dotted curve as θ varies from 180° to 360°. The

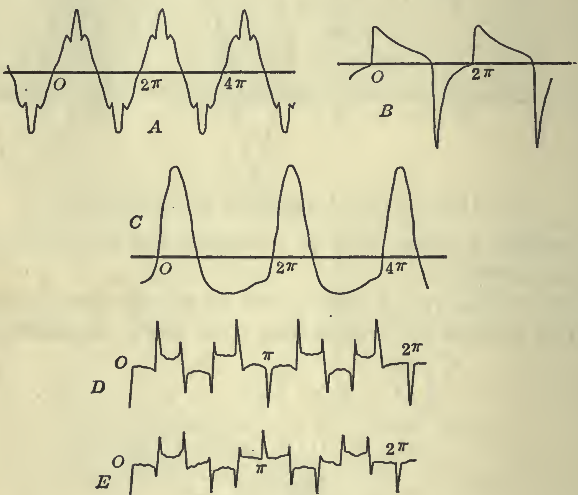


Fig. 146.—Curves made up of Odd Harmonics only, of Even Harmonics only, and of Both.

numbered points 1, 2, 3, 4, . . . of Fig. 145 correspond to the similarly numbered points in Fig. 144. The curves of Fig. 144 and Fig. 145 correspond, except that the constant term was omitted from the equation in constructing Fig. 145.

Exercises

1. If f be the frequency of the fundamental harmonic, show that:

$$y = \sin 2\pi ft + \sin 6\pi ft + \sin 10\pi ft + \ldots$$

contains odd harmonics only.

2. Write an expression containing even harmonics only, using the frequency f as in the last exercise.

3. How many loops has:

$$\rho = \cos 5\theta; \qquad \rho = \sin 6\theta;$$

$$\rho = \cos 7\left(\theta - \frac{\pi}{6}\right); \qquad \rho = \sin 4\theta?$$

4. In the diagram, Fig. 146, pick out curves made up of odd harmonics only, of even harmonics only, and those made up of both odd and even harmonics.

218.* Simple Periodic Variation Represented by a Complex Number. Fluctuating magnitudes exist that follow the *law* of S.H.M. although, strictly speaking, such magnitudes can be said to be "simple harmonic motions" in only a figurative sense. For example we may think of the fluctuations of the voltage or amperage in an alternating current as following such a law. Thus if E represent the electromotive force or pressure of the alternating current, then the fluctuations are expressed by

$$E = E_0 \sin \omega t$$

or by

$$E = E_0 \sin 2\pi f t$$

where f is the frequency of the fluctuation. Instead of S.H.M. such a variable is more accurately called a **sinusoidal varying magnitude,** although for brevity we shall often call it S.H.M. The graph in rectangular coördinates of such a periodic function is often called a "wave," although this term should, in exact language, be reserved for *a moving periodic curve,* such as $y = a \sin (hx - kt)$.

If the polar representation

$$\rho = a \sin \omega(t - t_1) \tag{1}$$

of the sinusoidal varying magnitude be used, then, as noted in the last section, the graph of (1) is a circle of diameter a inclined the angular amount ωt_1 to the left of the axis OY, as is seen at once by calling $\omega t = \theta$ and $\omega t_1 = \alpha$ in the equation of the circle $\rho = a \sin (\theta - \alpha)$. The circle can be drawn when the length and direction of its diameter are known; that is, the circle is completely specified when a and the direction of a (told by α) are given. Therefore the simple harmonic motion is completely symbolized by a *vector OA* of length a drawn from the origin in the direction given by the angle ωt_1; the direction angle of the vector OA is $\alpha + \frac{\pi}{2}$ or $\omega t_1 + \frac{\pi}{2}$.

The circle on the vector OA is located or characterized equally well if the rectangular coördinates (c, d) of the end of the diameter of the circle be given. But the complex number $c + di$ is represented by a vector which coincides with the diameter a of this circle. Hence we may represent the circle by the complex number $c + di$. Its modulus is $a = \sqrt{c^2 + d^2}$ and its amplitude is $\alpha + \frac{\pi}{2}$. Therefore if in (1) we take $a = \sqrt{c^2 + d^2}$, $\omega t_1 = \alpha$ and the variable angle $\omega t = \theta$, we can completely determine the S.H.M. by the complex number $c + id$. In the theory of alternating currents the sinusoidal varying current or voltage can conveniently be represented by a complex number, and that method of representing such magnitudes is in common use.

One of the advantages of representing S.H.M. by a vector or by a complex number is the fact that two or more such motions of like periods may then be compounded by the law of addition of vectors. This method of finding the resultant of two sinusoidal varying magnitudes of like periods possesses remarkable utility and simplicity.

To summarize, we may say:

(*a*) *A sinusoidal varying magnitude is represented graphically in polar coördinates by a vector, which by its length denotes the amplitude and by its direction angle with respect to OY denotes the epoch angle.*

(*b*) *Sinusoidal varying magnitudes of like periods may be compounded or resolved graphically by the law of parallelogram of vectors.*

If two sinusoidal varying magnitudes of like periods are in quadrature (that is, if their epoch angles differ by 90°), their relation, neglecting their epochs, can be completely expressed by a single complex number. Thus let two S.H.M. in quadrature

$$E_o = 113 \sin \omega(t - t_1) \tag{2}$$

and

$$E_c = 40 \cos \omega(t - t_1) \tag{3}$$

be represented by the circles and by the vectors marked OE_o and OE_c, Fig. 147. Call the resultant of these E_i. Then

$$E_i = 113 \sin \omega(t - t_1) + 40 \cos \omega(t - t_1) \tag{4}$$
$$= \sqrt{40^2 + 113^2} \sin \omega(t - t_2)$$
$$= 120 \sin \omega(t - t_2) \tag{5}$$

where ωt_2 is measured as shown in Fig. 148. Instead of representing (2) and (3) in the polar diagram by OE_o and OE_c and their

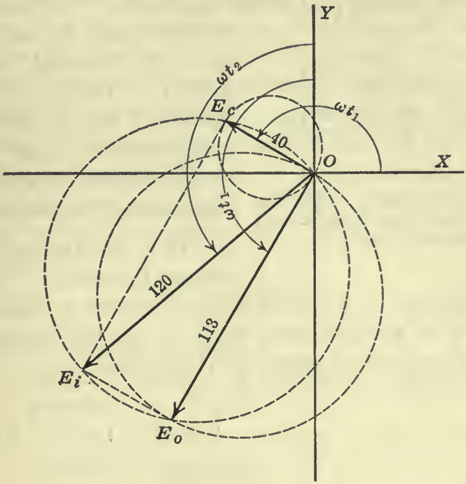


FIG. 147.—Composition of Two S.H.M. in Quadrature by Law of Addition of Vectors.

resultant by OE_i, we may represent (2), (3) and (4) in the complex number diagram, Fig. 148, by E_o, iE_c and $E_o + iE_c$, respectively. Since the modulus and amplitude of $E_o + iE_c$ are $\sqrt{E_o^2 + E_c^2}$ and α, respectively, and since the epoch angle of the resultant in Fig. 147 is $\omega t_2 = \omega t_1 - \alpha$, we can state the resultant as follows:

If we have given two S.H.M.'s in quadrature and take the amplitude of the one possessing the greater epoch angle as c and the amplitude of the other S.H.M. as d, and construct the complex number c + di, then this complex number c + di completely characterizes both of the S.H.M's. and their resultant. For, we can determine the modulus ρ and the amplitude α of $c + di$ and then if ωt_1 is the epoch angle of the motion with amplitude c, the epoch angle of the resultant is $\omega t_1 - \alpha$.

If we consider the two harmonic motions:

$$\rho = a_1 \sin \omega(t - t_1)$$

and

$$\rho = a_2 \sin \omega(t - t_2)$$

then if t_1 be greater than t_2 the first S.H.M. reaches its maximum value *after* the second reaches its maximum. The first S.H.M. is therefore said to *lag* the amount $(t_1 - t_2)$ behind the second S.H.M. That is, a S.H.M. represented by a circle located anticlockwise from a second circle represents a S.H.M. that *lags* behind the second.

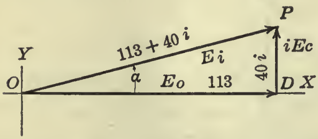


FIG. 148.—Complex Number Representation of the facts shown by Polar Diagram, Fig. 147.

219.* Illustration from Alternating Currents. The steady current C flowing in a simple electric circuit is determined by the pressure or electromotive force E and the resistance R according to the equation known as Ohm's law:

$$C = \frac{E}{R}$$

or,

$$E = CR$$

E is the pressure or voltage required to make the current C flow against the resistance R. If the current, instead of being steady, varies or fluctuates, then the pressure CR required to make the current C flow over the true resistance is called the **ohmic voltage** or ohmic pressure. But a changing or fluctuating current in an inductive circuit sets up a changing magnetic field around the circuit, from which there results a counter electromotive force or choking effect due to the changing of the current strength. This electromotive force is called the **reactive voltage** or **reactive pressure.** The choking effect that it has on the current is known as the **inductive reactance.** In case of a periodically changing current it acts alternately with and against the current. Opposite to the reactive voltage there is a component of the impressed voltage that is *consumed* by the reactance. See Fig. 149.

The pressure which is at every instant applied to the circuit from without is called the **impressed electromotive force** or voltage. Of the three pressures—namely, the impressed voltage, the ohmic voltage (consumed by the resistance) and the reactive voltage consumed by the inductive reactance, any one may always be regarded as the resultant of the other two. Hence if in a polar diagram the pressures be represented in magnitude and relative phase by the sides of a parallelogram, the impressed voltage may be regarded as the diagonal of a parallelogram of which the other two pressures are sides. Since, however, the reactance or the counter inductive pressure depends upon the rate of change of the current, it lags, in the case of a sinusoidal current, 90° behind the true or ohmic voltage, which last is always in phase with the current. The pressure consumed by the counter inductive pressure therefore leads the current by 90°. Thus, in the language of complex numbers

$$E_i = E_o + iE_c \tag{1}$$

in which

E_i = impressed pressure

E_o = ohmic pressure, or pressure consumed by the resistance

E_c = counter inductive pressure, or the pressure consumed by reactance

It is found that the counter inductive pressure depends upon a constant of the circuit L called the **inductance** and upon the *angular velocity* or frequency of the alternating impressed pressure, so that:

$$E_c = 2\pi fLC = \omega LC$$

Hence (1) may be written:

$$E_i = RC + i2\pi fLC \tag{2}$$

$$= RC + i\omega LC \tag{3}$$

The modulus of the complex number on the right of this equation is

$$C\sqrt{R_2 + \omega^2L^2}$$

Considering, then, merely the absolute value $|E_o|$ and $|C|$ of pressure and current, we may write:

$$|C| = \frac{|E_i|}{\sqrt{R^2 + \omega^2 L^2}} \tag{4}$$

From the analogy of this to Ohm's law:

$$C = \frac{E}{R}$$

the denominator $\sqrt{R^2 + \omega^2 L^2}$ is thought of as limiting or restricting the current and is called the **impedance** of the circuit.

Let there be a condenser in the circuit of an alternator, but let the circuit be free from inductance. Then besides the pressure consumed by the resistance, an additional pressure is required at any instant to hold the charge on the condenser. If K be the capacity of the condenser, it is found that that part of the pressure consumed in holding the charge on the condenser is $\frac{C}{2\pi f K}$ or $\frac{C}{\omega K}$ and is in phase position 90° behind the current C. The choking effect of this on the current may be called the **condensive reactance**. When a condenser is in the circuit *in addition to inductance,* the total pressure consumed by the reactance has the form:

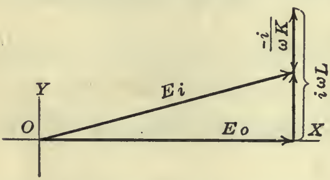


Fig. 149.—Complex Number Diagram of Equation 5, §219

$$2\pi f L - \frac{1}{2\pi f K}$$

and the complex number that symbolizes the vector is

$$E_i = RC + i2\pi f CL - \frac{iC}{2\pi f K} \tag{5}$$

(see Fig. 149).

Further illustrations of the applications of complex numbers to alternating currents is out of place in this book. The illustrations are merely for the purpose of emphasizing the usefulness of these numbers in applied science. An interesting application of the use of complex numbers to the problem of the steam turbine will be found in Steinmetz's "Engineering Mathematics," page 33.

Exercises

1. Draw the polar diagram and complex number representation of E_i if $R = 5$, $C = 21$, $f = 60$, $L = 0.009$, $K = 0.005$.

2. Draw a similar diagram if $E_i = 100$, $E_o = 90$, $f = 40$, $L = 0.008$, $K = 0.003$.

220. Product of Complex Numbers. *The product of two or more complex numbers is a complex number whose modulus is the product of the moduli and whose amplitude is the sum of the amplitudes of the complex numbers.* Let the complex numbers be:

$$z_1 = x_1 + y_1 i$$
$$= r_1 (\cos \theta_1 + i \sin \theta_1)$$
$$z_2 = x_2 + y_2 i$$
$$= r_2 (\cos \theta_2 + i \sin \theta_2), \text{ etc.}$$

By actual multiplication, we get:

$$z_1 z_2 = r_1 r_1 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) i]$$
$$= r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)]$$

Whence it is seen that $r_1 r_2$ is the modulus of the product and $(\theta_1 + \theta_2)$ is the amplitude.

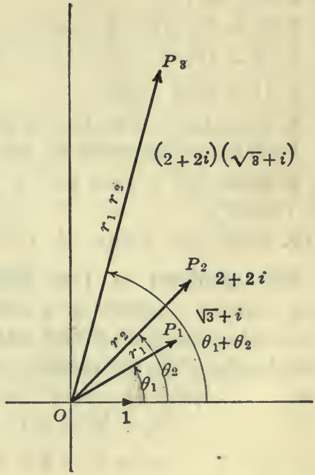


FIG. 150.—Product of Two Complex Numbers.

The above theorem is illustrated by Fig. 150. If the two given complex numbers be represented by their vectors OP_1 and OP_2, their product will be represented by the vector OP_3 whose direction angle is the sum of the amplitudes of the two given factors, and whose length OP_3 is the product of the lengths OP_1 and OP_2.

The figure represents the product $(2 + 2i)(\sqrt{3} + i)$. Expressed in terms of modulus and amplitude these may be written:

$$\sqrt{3} + i = 2(\cos 30° + i \sin 30°)$$
$$2 + 2i = 2\sqrt{2}(\cos 45° + i \sin 45°)$$

Hence $r_1 = 2$, $r_2 = 2\sqrt{2}$, $\theta_1 = 30°$, $\theta_2 = 45°$

Therefore: $(2 + 2i)(\sqrt{3} + i) = 4\sqrt{2} (\cos 75° + i \sin 75°)$

Exercises

Find the moduli and amplitudes of the following products, and construct the factors and products graphically. Take a positive angle for the amplitude in every case.

1. $(1+\sqrt{3}i)(2\sqrt{3}+2i)$.

2. $(2+\frac{2}{3}\sqrt{3}\,i)(2+2i)$.

3. $(\sqrt{3}+3i)(2-2i)$.

4. $(1+i)^2$.

5. $(2-2\sqrt{3}i)(\sqrt{3}+3i)$.

6. $(1-i)^4$.

7. $(1+i)^2(1-i)^2$.

8. $2(\cos 15° + i \sin 15°) \times 3(\cos 25° + i \sin 25°)$.

Find numerical result by use of slide rule or trigonometric tables.

9. $2(\cos 10° + i \sin 10°) \times (1/3)(\cos 12° + i \sin 12°) \times 6(\cos 8° + i \sin 8°)$.

10. Find the value of $4\sqrt{2}(\cos 75° + i \sin 75°) \div (\sqrt{3}+i)$.

221. Quotient of Two Complex Numbers. *The quotient of two complex numbers is a complex number whose modulus is the quotient of the moduli and whose amplitude is the difference of the amplitudes of the two complex numbers.* Let the complex numbers be:

$$z_1 = x_1 + y_1 i = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = x_2 + y_2 i = r_2(\cos\theta_2 + i\sin\theta_2)$$

We have:

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$
$$= \frac{r_1[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}$$
$$= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

Whence it is seen that $\frac{r_1}{r_2}$ is the modulus of the quotient and $(\theta_1 - \theta_2)$ is the amplitude.

In Fig. 151, the complex number represented by the vector OP_1 when divided by the complex number represented by OP_2 yields the result represented by OP_3, whose length r_1/r_2 is found by dividing the length of OP_1 by the length of OP_2, and whose direction angle is the difference $(\theta_1 - \theta_2)$ of the amplitudes of OP_1 and OP_2. The figure is drawn to scale for the case:

$$\frac{5(\cos 60^\circ + i \sin 60^\circ)}{2(\cos 20^\circ + i \sin 20^\circ)} = (2.5)\ (\cos 40^\circ + i \sin 40^\circ)$$

Exercises

Find the quotient and graph the result in each of the following exercises. Always take amplitudes as positive angles and if $\theta_2 < \theta_1$, take $\theta_1 + 360^\circ$ instead of θ_1.

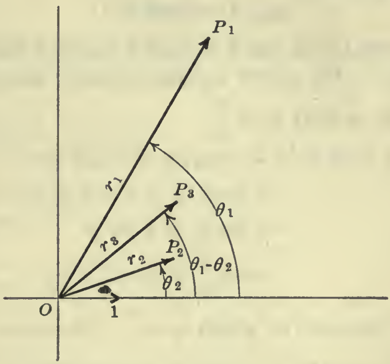


FIG. 151.—Quotient of Two Complex Numbers.

1. $(1 + \sqrt{3}i) \div (2 + \sqrt{2}i)$.
2. $(\frac{1}{2} + \frac{1}{2}\sqrt{3}i) \div (\sqrt{2} - \sqrt{2}i)$.
3. $(3\sqrt{3} - 3i) \div (-1 + \sqrt{3}i)$.
4. $(1 - \sqrt{3}i) \div i$.
5. $2(\cos 36^\circ + i \sin 36^\circ) \div 5(\cos 4^\circ + i \sin 4^\circ)$.
6. $12(\cos 48^\circ + i \sin 48^\circ) \div [2(\cos 15^\circ + i \sin 15^\circ)$
$3(\cos 9^\circ + i \sin 9^\circ)]$.
7. $\dfrac{[4 + (4/3)\sqrt{3}\,i]\,(2 + 2\sqrt{3}\,i)}{8 + 8i}$.
8. Express in terms of a, b, c, d the amplitude of $(a + bi) \div (c + di)$.

222. De Moivre's Theorem. As a special case of §220 consider the expression:

$$(\cos\theta + i\sin\theta)^n$$

This being the product of n factors like $(\cos\theta + i\sin\theta)$, we write, by means of **§220**:

$$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)\ \ldots$$
$$= [\cos(\theta + \theta + \ldots) + i\sin(\theta + \theta + \ldots)]$$

or:

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta) \tag{1}$$

which relation is known as De Moivre's theorem.

De Moivre's theorem holds for fractional values of n. For, first consider the expression:

$$(\cos\theta + i\sin\theta)^{1/t}$$

where the power $1/t$ of $\cos\theta + i\sin\theta$ is, by definition, an operator such that its tth power equals $\cos\theta + i\sin\theta$.

Put $\theta = t\phi$, so that $\phi = \dfrac{\theta}{t}$

Then: $(\cos\theta + i\sin\theta)^{1/t} = (\cos t\phi + i\sin t\phi)^{1/t}$

$$= [(\cos\phi + i\sin\phi)^t]^{1/t} \text{ by (1)}$$
$$= \cos\phi + i\sin\phi$$
$$= \cos\frac{\theta}{t} + i\sin\frac{\theta}{t} \tag{2}$$

Next consider the case in which $n = \dfrac{s}{t}$. We know:

$$(\cos\theta + i\sin\theta)^{s/t} = [(\cos\theta + i\sin\theta)^s)]^{1/t}$$
$$= (\cos s\theta + i\sin s\theta)^{1/t} \text{ by (1)}$$
$$= \cos\frac{s\theta}{t} + i\sin\frac{s\theta}{t} \text{ by (2)} \tag{3}$$

Likewise the theorem may be proved for negative values of n.

The following examples illustrate the application of De Moivre's theorem.

(1) Find $(3 + i\sqrt{3})^4$.

write: $$3 + i\sqrt{3} = 2\sqrt{3}(\cos 30° + i\sin 30°)$$

Then, by De Moivre's theorem:

$$(3 + i\sqrt{3})^4 = 144(\cos 120° + i\sin 120°)$$
$$= 144\left(-1/2 + \tfrac{1}{2}\sqrt{3}\,i\right)$$
$$= -72 + 72\sqrt{3}\,i$$

(2) Find $(2 + 2i)^{11}$.

Write: $2 + 2i$ in the form:

$$2 + 2i = 2\sqrt{2}(\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}i)$$

$$\begin{aligned}(2 + 2i)^{11} &= (2\sqrt{2})^{11}(\cos 45° + i \sin 45°)^{11} \\ &= (2\sqrt{2})^{11}(\cos 495° + i \sin 495°) \\ &\doteq (2\sqrt{2})^{11}(\cos 135° + i \sin 135°) \\ &= (2\sqrt{2})^{11}(-\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}i) \\ &= 2^{16}(-1 + i)\end{aligned}$$

Exercises

Evaluate the following by De Moivre's theorem, using trigonometric table or slide rule when necessary.

1. $(8 + 8\sqrt{3}i)^{12}$.

2. $\sqrt[3]{27(\cos 75° - i \sin 75°)}$.

3. $\sqrt[3]{125\, i}$.

4. $[\cos 9° + i \sin 9°]^{10}$.

5. $(3 + \sqrt{3}i)^5$.

6. $[1/2 + (1/2)\sqrt{3}i]^4$.

7. $(1 + i)^8$.

8. $(-2 + 2i)^{1/3}$.

9. $[(1/2)\sqrt{3} - (1/2)i]^5$.

10. Find value of $(-1 + \sqrt{-3})^5 + (-1 - \sqrt{-3})^5$ by De Moivre's theorem.

11. Find the value of $x^2 - 2x + 2$ for $x = 1 + i$.

12. If $j_1 = -1/2 + (1/2)\sqrt{-3}$ and $j_2 = -1/2 - (1/2)\sqrt{-3}$, show that $j_1^3 = 1$, $j_2^3 = 1$, $j_1^2 = j_2$, $j_2^2 = j_1$, $j_1^{3n} = j_2^{3n} = 1$, $j_1^{3n+1} = j_1$.

223. The Roots of Unity. Unity may be written:

$$\begin{aligned}1 &= \cos 0 + i \sin 0 \\ 1 &= \cos 2\pi + i \sin 2\pi \\ 1 &= \cos 4\pi + i \sin 4\pi \\ 1 &= \cos 6\pi + i \sin 6\pi \\ &\cdots\cdots\end{aligned}$$

and so on. By De Moivre's theorem the cube root of any of these is taken by dividing the amplitudes by 3. Therefore, from the above expressions in turn there results:

$$\sqrt[3]{1} = \cos 0 + i \sin 0 = 1$$

$$\sqrt[3]{1} = \cos(2\pi/3) + i\sin(2\pi/3) = \cos 120° + i \sin 120°$$
$$= -1/2 + i(1/2)\sqrt{3}$$

$$\sqrt[3]{1} = \cos(4\pi/3) + i\sin(4\pi/3) = \cos 240° + i\sin 240°$$
$$= -1/2 - i(1/2)\sqrt{3}$$

$$\sqrt[3]{1} = \cos 6\pi/3 + i \sin 6\pi/3 = \text{same as first, etc.}$$

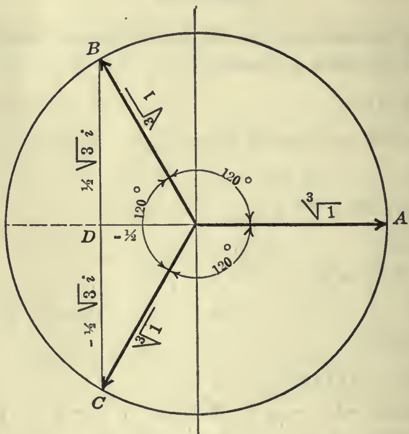


FIG. 152.—The Cube Roots of Unity.

Therefore there are three cube roots of unity. Since these are the roots of the equation $x^3 - 1 = 0$, they might have been found by factoring, thus:

$$x^3 - 1 = (x-1)(x^2 + x + 1)$$
$$= (x-1)(x + 1/2 + \tfrac{1}{2}\sqrt{3}\,i)(x + 1/2 - \tfrac{1}{2}\sqrt{3}\,i)$$

The three roots of unity divide the angular space about the point O into three equal angles, as shown in Fig. 152. In the same way, it can be shown that there are four fourth roots, five fifth roots, etc., of unity and that the vectors representing them have modulus 1 and amplitudes that divide equally the space about O.

To find all of the roots of any complex number, proceed as in the following illustrative examples.

(1) Find $\sqrt{\sqrt{3} + 3i}$.

Write $\sqrt{3} + 3i$ in the form:

$$\sqrt{3} + 3i = 2\sqrt{3}(\cos 60° + i \sin 60°)$$

Hence, by De Moivre's theorem:

$$\begin{aligned}(\sqrt{3} + 3i)^{1/2} &= \sqrt[4]{12}\,(\cos 30° + i \sin 30°)\\ &= \sqrt[4]{12}\,[(1/2)\,\sqrt{3} + (1/2)i]\\ &= (1/2)\sqrt[4]{108} + (1/2)\sqrt[4]{12}\,i\end{aligned}$$

A second root can be found by writing:

$$\sqrt{3} + 3i = 2\sqrt{3}[\cos (60° + 360°) + i \sin (60° + 360°)]$$

since adding a multiple of 360° to the amplitude does not change the value of the sine and cosine. In applying De Moivre's theorem there results:

$$\begin{aligned}(\sqrt{3} + 3i)^{1/2} &= \sqrt[4]{12}\,(\cos 210° + i \sin 210°)\\ &= \sqrt[4]{12}[-(1/2)\sqrt{3} - (1/2)i]\end{aligned}$$

(2) Find the cube root of $-\sqrt{2} + \sqrt{2}i$.

We write:

$$\begin{aligned}-\sqrt{2} + i\sqrt{2} &= 2(\cos 135° + i \cos 135°)\\ &= 2[\cos (135° + n360°) + i \cos (135° + n360°)]\end{aligned}$$

in which n is any integer. Hence:

$$\begin{aligned}(-\sqrt{2} + i\sqrt{2})^{1/3} &= \sqrt[3]{2}[\cos (45° + n120°) + i \sin (45° + n\,120°)]^{1/3}\\ &= \sqrt[3]{2}(\cos 45° + i \sin 45°) \quad \text{for } n = 0\\ &= \sqrt[3]{2}(\cos 165° + i \sin 165°) \text{ for } n = 1\\ &= \sqrt[3]{2}(\cos 285° + i \sin 285°) \text{ for } n = 2\end{aligned}$$

These are the three cube roots of the given complex number. For $n = 3$ the first root is obtained a second time.

Exercises

Find all the indicated roots of the following:

1. $(8 + 8\sqrt{3}\,i)^{1/2}$.
2. $\sqrt[3]{27(\cos 75° - i \sin 75°)}$.
3. $\sqrt[3]{125i}$.

4. $(-2 + 2i)^{1/5}$.

5. $(2 + 2i)^{1/3}$.

6. $32^{1/3}$.

7. $\sqrt[9]{512}$.

8. Find to four places one of the imaginary 7th roots of $+1$.
NOTE: $\cos 51° 25.7' + i \sin 51° 25.7' = 0.6235 + 0.7818\, i$.

224. Inverse Functions. The exponent (-1) attached to a symbol of operation signifies the "undoing" of the operation denoted by the symbol of operation. The number of different operations in mathematics is an *even number;* that is, for every operation we define, we may, and usually do, define the operation that "undoes" the given operation. Thus if we define *addition,* we at once follow it by defining the undoing of addition, or *subtraction;* if we define multiplication, we follow it with the concept of the undoing of multiplication, or *division;* if we define involution, or the raising to powers, we also define the undoing of this operation, namely *evolution,* or the extraction of roots. The second of each of these pairs of operations is called the **inverse** of the first operation, and *vice versa.*

The exponent (-1) attached to any symbol of operation is defined to mean the inverse of the operation called for by the symbol to which it is attached. Thus 2^{-1} is not a *doubler;* the operation called for is the "*undoing*" of doubling, or *halving.* The symbol $\log^{-1} x$, read the "anti-logarithm of x," calls for the number of which x is the logarithm. Thus, if the base be 10, $\log^{-1} 2 = 100$, $\log^{-1} 3 = 1000$, $\log^{-1} 0.3010 = 2$, $\log^{-1} 1 = 10$, $\log^{-1} 0 = 1$, etc. Since "$\log^{-1}$" is the symbol of undoing the operation indicated by "log," the double symbol $(\log^{-1} \log)$ must leave the operand unchanged. The operator that leaves an operand unchanged is unity. Hence a double symbol like $(\log^{-1} \log)$ can always be replaced by 1; thus $\log^{-1} \log 467 = 467$; also $\log \log^{-1} 467 = 467$.

$$\text{Likewise } 3^{-1}\cdot 3\cdot 1 = 1;\ (\sqrt{3})^{-1}\cdot(\sqrt{3})\cdot 1 = 1, \text{ etc.}$$

An important use of the present notation is in the symbols $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc., used in **§70.** These are read "anti-sine of x," "anti-cosine of x," etc., or "the angle whose sine is x," "the angle whose cosine is x," etc. Thus $\sin^{-1} (1/2) = 30°$, $\tan^{-1}$

$= 45°$, $\cos^{-1} 0 = \pi/2$, etc. Note that $\log^{-1} x$ must be carefully distinguished from $(\log x)^{-1}$, which means $1/\log x$; similarly, $\sin^{-1} x$ must be distinguished from $(\sin x)^{-1}$. A notation like $\log x^{-1}$ is ambiguous, and should never be used.

If we write $r = \cos\theta$, $y = \log x$, $y = \tan x$, the same functional relations may be expressed in the inverse notation by $\theta = \cos^{-1} r$, $x = \log^{-1} y$, $x = \tan^{-1} y$. Thus $y = a^x$, $x = \log_a y$, $y = \log_a^{-1} x$, and $y = \exp_a x$ are four ways of expressing the same relation between x and y.

Any relation expressed by means of the direct functions may also be expressed in terms of the inverse functions. Thus we know:

$$\log (xy) = \log x + \log y \tag{1}$$

Let $\log x = a$, $\log y = b$, then it follows that:

$$x = \log^{-1} a,\ y = \log^{-1} b$$

Hence (1) becomes:

$$\log (\log^{-1} a \log^{-1} b) = a + b$$

or:

$$\log^{-1} a \log^{-1} b = \log^{-1} (a + b) \tag{2}$$

Likewise consider:

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \tag{3}$$

Let $\quad \sin \alpha = a$ and $\sin \beta = b$

then: $\quad \alpha = \sin^{-1} a,\ \beta = \sin^{-1} b$

Also since

$$\sin \alpha = a$$

$$\cos \alpha = \sqrt{1 - a^2}$$

Likewise:

$$\cos \beta = \sqrt{1 - b^2}$$

Hence (3) may be written:

$$\sin (\sin^{-1} a + \sin^{-1} b) = a\sqrt{1 - b^2} + b\sqrt{1 - a^2}$$

or:

$$\sin^{-1} a + \sin^{-1} b = \sin^{-1} (a\sqrt{1 - b^2} + b\sqrt{1 - a^2})$$

Since there are many angles whose sine is equal to a given number x, it is desirable to specify by definition which angle is meant. The following conventions are therefore useful:

$\sin^{-1} x$ means the angle between $-90°$ and $+90°$ whose sine is x.

$\cos^{-1} x$ means the angle between 0° and 180° whose cosine is x.

$\tan^{-1} x$ means the angle between $-90°$ and $+90°$ whose tangent is x.

Exercises

1. Show that $\sin^{-1}(1/2) + \sin^{-1}\sqrt{1/2} = 5\pi/12$.

2. Show that $\sin^{-1} x + \cos^{-1} x + \cos^{-1} x = \pi/2$.

3. Is there any difference between the graph of $y = f(x)$ and the graph of $x = f^{-1}(y)$?

4. Prove that $\tan^{-1} x + \tan^{-1}(1/x) = \pi/2$.[1]

5. Find the value of x in the equation $\sin^{-1} x + \sin^{-1} 2x = \pi/3$.

6. If $f(x) = x^5$, find $f^{-1}(x)$.

Let $y = f(x) = x^5$. Then $x = f^{-1}(y) = \sqrt[5]{y}$. Hence if $f^{-1}(y) = \sqrt[5]{y}$, then $f^{-1}(x) = \sqrt[5]{x}$.

7. If $f(x) = e^x$, find $f^{-1}(x)$.

8. What is the inverse of $f(\theta) = 1 - \theta$?

Let $y = f(\theta)$, so that $\theta = f^{-1}(y)$, etc.

9. Show that the function

$$y = \frac{x+1}{x-1}$$

is its own inverse.

[1] The symbol ($=$) may here be interpreted as meaning "congruent to."

CHAPTER XII

LOCI

225. Parametric Equations. The equation of a plane curve is ordinarily given by an equation in two variables, as has been amply illustrated by numerous examples in the preceding chapters. It is obvious that a curve might also be given by two equations containing three variables, for if the third variable be eliminated from the two equations, a single equation in two variables results. When it is desirable to describe a locus by means of two equations in three variables the equations are known as **parametric equations,** as has already been explained in **§74.** Two of the variables usually belong to one of the common coördinate systems and the third is an extra variable called the *parameter.* In applied science the variable *time* frequently occurs as a parameter.

The parametric equations of the circle have already been written. They are:

$$\begin{aligned} x &= a \cos \theta \\ y &= a \sin \theta \end{aligned} \qquad (1)$$

where the parameter θ is the direction angle of the radius vector to the point (x, y). Likewise the parametric equations of the ellipse have been written:

$$\begin{aligned} x &= a \cos \theta \\ y &= b \sin \theta \end{aligned} \qquad (2)$$

and those of the hyperbola have been written:

$$\begin{aligned} x &= a \sec \theta \\ y &= b \tan \theta \end{aligned} \qquad (3)$$

In harmonic motion, the ellipse was seen to be the resultant of the two S.H.M. in quadrature:

$$\begin{aligned} x &= a \cos \omega t \\ y &= b \sin \omega t \end{aligned} \qquad (4)$$

Here the parameter t is *time.*

226. Problems in Loci. It is frequently required to find the equation of a locus when a description of the process of its generation is given in words, or when a mechanism by means of which the curve is generated is fully described. There is only one way to gain facility in obtaining the equations of curves thus described, and that is by the solution of numerous problems. Sometimes it is best to seek the parametric equations of the curve, but sometimes the ordinary polar or Cartesian equation can be obtained directly. The following problems are illustrative:

(1) A straight line of constant length $a + b$ moves with its ends always sliding on two fixed lines at right angles to each other. Find the equation of the curve described by any point of the moving line. (See §**75**.)

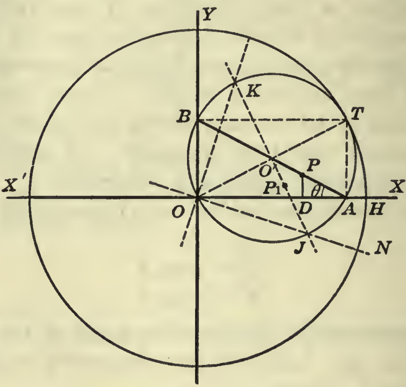


FIG. 153.—Generation of So-called "Elliptic Motion."

In Fig. 153, let AB be the line of fixed length, and let it so move that A remains on the x-axis and B remains on the y-axis. Let any point of this line be P whose distance from A is b and whose distance from B is a. If the angle $X'AB$ be called θ, then PD, the ordinate of P, is:

$$y = b \sin \theta$$

and OD, the abscissa of P, is:

$$x = a \cos \theta$$

Therefore P describes an ellipse of semi-axes a and b.

(2) A circle rolls without slipping within a circle of twice the diameter. Show that any point attached to the moving circle[1] describes an ellipse.

[1] "Circle" is here used in the sense of a "disc" or circular area and not in the sense of a "circumference."

Draw the smaller rolling circle in any position within the larger circle, and call the point of tangency T, as in Fig. 153. Since the smaller circle is half the size of the larger circle, the smaller circle always passes through O, and the line joining the points of intersection of the small circle with the coördinate axes is, for all positions, a diameter, since the angle AOB is a right angle.

If we can prove that the arc AT = the arc HT for all positions of T, then we shall have shown that as the small circle rolls from an initial position with point of contact at H, the end A of the diameter AB slides on the line OX. Since B lies on OY and since AB is of fixed length, this proves by problem (1) that any point of the small circle lying on the particular diameter AB describes an ellipse.

To prove that arc AT = arc HT, we have that the angle HOT is measured in radians by $\frac{\text{arc } HT}{OH}$. The angle $AO'T$ is measured in radians by $\frac{\text{arc } AT}{O'A}$. Since $\angle AO'T = 2\angle HOT$, we have

$$\frac{\text{arc } AT}{O'A} = 2\,\frac{\text{arc } HT}{OH}$$

But, $OH = 2O'A$. Hence arc AT = arc HT.

We can now prove that any other point of the rolling circle describes an ellipse. Let any other point be P_1. Through P_1 draw the diameter $JO'K$. The above reasoning applies directly, replacing A by J and H by N.

It is easy to see that all points equidistant from the center such as the points P, P_1, of the small circle, describe ellipses of the same semi-axes a and b, but with their major axes variously inclined to OH.

(3) Determine the curve given by the parametric equations:

$$x = a \cos 2\omega t \tag{1}$$
$$y = a \sin \omega t \tag{2}$$

To eliminate t, the first equation may be written

$$x = a\,(1 - 2 \sin^2 \omega t) \tag{3}$$

From the second equation, $\sin \omega t = \frac{y}{a}$. Substituting for $\sin \omega t$ in (3),

$$x = a\left(1 - \frac{2y^2}{a^2}\right) \tag{4}$$

or,

$$y^2 = -\frac{a}{2}x + \frac{a^2}{2} \tag{5}$$

This curve is the parabola $y^2 = mx$, the special location of which the student should describe.

(4) Construct a graph such that the *increase* in y varies directly as x.

If y varied directly as x, then y would equal kx, where k is any constant. In the given problem the *increase* in y (and not y itself) must vary in this manner. Let the initial value of y be represented by y_0. Then the gain or increase of y is represented by $y - y_0$. Hence, by the problem:

$$y - y_0 = kx \tag{1}$$

Since y_0 is a constant, (1) is the equation of the straight line of slope k and intercept on the y-axis $= y_0$, which ordinarily would be written in the form:

$$y = kx + y_0$$

(5) Express the diagonal of a cube as a function of its edge, and graph the function.

If the edge of the cube be x, its diagonal is $\sqrt{x^2 + x^2 + x^2}$ or $x\sqrt{3}$. If the diagonal be represented by y, we have $y = \sqrt{3}x$, which is a straight line.

(6) A rectangle whose length is twice its breadth is to be inscribed in a circle of radius a. Express the area of this rectangle in terms of the radius of the circle.

Let the rectangle be drawn in a circle whose equation is $x^2 + y^2 = a^2$. At a corner of the rectangle we have $x = 2y$. The area A of the rectangle is $4xy$, or since $x = 2y$, is $8y^2$. From the equation of the circle we obtain $4y^2 + y^2 = a^2$ or $y^2 = a^2/5$. Hence:

$$A = (8/5)a^2$$

If A and a be graphed as Cartesian variables, the graph is a parabola.

(7) A rectangle is inscribed in a circle. Express its area as a function of a half of one side.

Here, as above:

$$A = 4xy = 4x\sqrt{a^2 - x^2}$$

The student should graph this curve, for which purpose a may be put equal to unity. First draw the semicircle $y = \sqrt{a^2 - x^2}$. For $x = 1/5$ take one-fifth of the ordinate of this semicircle. For $x = 2/5$ take two-fifths of the ordinate of the semicircle, and so on. The curve through these points is $y = x\sqrt{a^2 - x^2}$, from which $y = 4x\sqrt{a^2 - x^2}$ can be had by proper change in the vertical unit of measure.

Exercises

1. In polar coördinates, draw the curves:

$$r = 2 \cos \theta \qquad r = 2 \cos \theta + 1$$
$$r = 2 \cos \theta - 1 \qquad r = 2 \cos \theta + 3.$$

2. A curve (polar coördinates) passes through the point (1, 1). (This means the point whose coördinates are *one centimeter*, and *one radian.*) Starting at this point, a point moves so that the radius vector of the point is always equal to the vectorial angle. Sketch the curve. Write the polar equation of the curve.

3. A point moves so that one of its polar coördinates, the radius vector, varies directly as the other polar coördinate, the vectorial angle. Write the polar equation of such a curve. Does the curve go through the point (1, 1)?

4. A polar curve is generated by a point which starts at the point (1, 2) and moves so that the *increase* in the radius vector always equals the increase in the vectorial angle. Write the equation of the curve.

5. A polar curve is generated by a point which starts at the point (1, 2) and moves so that the increase in the radius vector varies directly as the increase in the vectorial angle. Write the equation of the curve.

6. A ball is thrown from a tower with a horizontal velocity of 10 feet per second. It falls at the same time through a variable distance given by $s = 16.1t^2$, where t is the elapsed time in seconds and s is in feet. Find the equation of the curve traced by the ball.

7. The point P divides the line AB, of fixed length, *externally* in the ratio $a:b$, that is, so that $PA/PB = a/b$. If the line AB move with its end points always remaining on two fixed lines OX and OY at right angles to each other, then P describes an ellipse of semi-axes a and b.

8. If in the last problem the lines OX and OY are not at right angles to each other, the point P still describes an ellipse.

9. A point moves so as to keep the ratio of its distances from two fixed lines AC and BD constant. Prove that the locus consists of four straight lines.

10. A sinusoidal wave of amplitude 6 cm. has a node at + 5 cm. and an adjacent crest at + 8 cm. Write the equation of the curve.

11. The velocity of a simple wave is 10 meters per second. The period is two seconds. Find the wave length and the frequency.

12. A polar curve passes through the point (1, 1) and the radius vector varies inversely as the vectorial angle. Plot the curve and write its equation. Consider especially the points where the vectorial angle becomes infinite and where it is zero. Sketch the same function in rectangular coördinates.

13. Rectangles are inscribed in a circle of radius r. Express by means of an equation and plot: (a) the area, and (b) the perimeter of the rectangles as a function of the breadth.

14. Right triangles are constructed on a line of given length h as hypotenuse. Express and plot: (a) the area, and (b) the perimeter as a function of the length of one leg.

15. A conical tent is to be constructed of given volume, V. Express and graph the amount of canvas required as a function of the radius of the base.

16. A closed cylindrical tin can is to be constructed of given volume, V. Plot the amount of tin required as a function of the radius of the can.

17. A rectangular water-tank lined with lead is to be constructed to hold 108 cubic feet. It has a square base and open top. Plot the amount of lead required as a function of the side of the base.

18. An open cylindrical water-tank is to be made of given volume, V. The cost of the sides per square foot is two-thirds the cost of the bottom per square foot. Plot the cost as a function of the diameter.

19. An open box is to be made from a sheet of pasteboard 12 inches square, by cutting equal squares from the four corners and bending up the sides. Plot the volume as a function of the side of one of the squares cut out.

20. The illumination of a plane surface by a luminous point varies directly as the cosine of the angle of incidence, and inversely as the square of the perpendicular distance from the surface. Plot the illumination of a point on the floor 10 feet from the wall, as a function of the height of a gas burner on the wall.

21. Using the vertical distances between corresponding points on the curves $y = \sin t$ and $y = -\sin t$ as ordinates and the vertical distances between corresponding points of $y = 2t$ and $y = t^2$ as abscissas, find the equation of the resulting curve.

227. Loci Defined by Focal Radii. A number of important curves are defined by imposing conditions upon the distances of any point of the locus from two fixed points, called **foci.**

(1) A point moves so that the product of its distances from two fixed points is constant. Find the equation of the path of the particle. Let the two fixed points F_1 and F_2, Fig. 154, be taken on the x-axis the distance a each side of the origin. Call the distances of P from

the fixed points r_1 and r_2. Then the variables r_1 and r_2 in terms of x and y are:

$$\begin{aligned} r_1^2 &= y^2 + (x - a)^2 \\ r_2^2 &= y^2 + (x + a)^2 \end{aligned} \tag{1}$$

Hence:

$$r_1^2 r_2^2 = [y^2 + (x - a)^2][y^2 + (x + a)^2] \tag{2}$$

Calling the constant value of $r_1 r_2 = c^2$, we have as the Cartesian equation of the locus:

$$[y^2 + (x - a)^2][y^2 + (x + a)^2] = c^4 \tag{3}$$

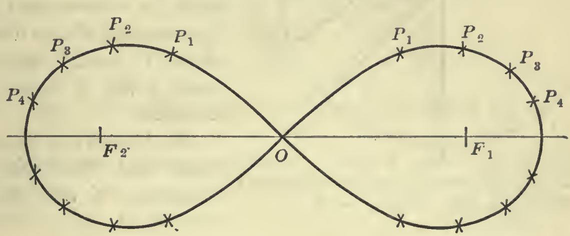


Fig. 154.—The Lemniscate.

which may be written:

$$(y^2 + x^2 + a^2)^2 - 4a^2 x^2 = c^4 \tag{4}$$

$$(x^2 + y^2)^2 + 2a^2x^2 + 2a^2y^2 + a^4 - 4a^2x^2 = c^4 \tag{5}$$

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2) + c^4 - a^4 \tag{6}$$

If $c = a$ the curve is called the **lemniscate,** and the Cartesian equation reduces to:

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2) \tag{7}$$

For other values of c the curves are known as the **Cassinian ovals.** When $c < a$ the curve consists of two separate ovals surrounding the foci, and for $c > a$ there is but a single oval. The curves are shown in Fig. 157. These curves give the form of the equipotential surfaces in a field around two positively or two negatively charged parallel wires. To construct the curves proceed as follows: In Fig. 155 let the circle have a radius c and in 156 let the circle have the diameter c.

In Fig. 155 we can use the theorem: "*If from a point without a circle a tangent and secant be drawn, the tangent is a mean proportional to the entire secant and the part without the circle.*" In Fig. 156 we can use the theorem: "*If from the vertex of the right angle of any right triangle a perpendicular be dropped upon the hypotenuse, then either* *leg of the triangle is a mean proportional between the hypotenuse and the adjacent segment."*

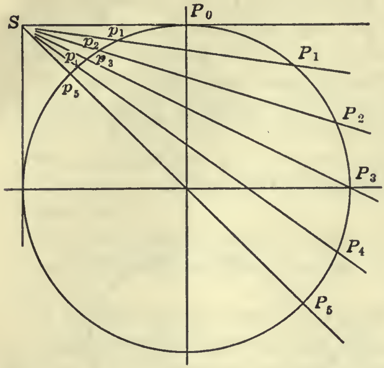


FIG. 155.—Construction of the Constant Products $SP_n \times Sp_n = c^2$.

Then in either Fig. 155 or 156, $SP_1 \times Sp_1 = c^2$ and likewise $SP_2 \times Sp_2 = c^2$, etc. Therefore SP_0, SP_1, SP_2, . . . are the values of r_2 that correspond to $r_1 = Sp_0$, Sp_1, Sp_2, . . . , respectively, and the ovals may be constructed by the intersection of arcs described about F_1 and F_2 as centers, using pairs of these values as radii.

(2) Construct the curve such that the ratio of the distances of any point of the curve from two fixed points is constant.

Let the two fixed points be A and B, Fig. 158; let the constant ratio of the distances of any point of the curve from the two fixed points be $r_1/r_2 = m/n$.

To find one point of the locus, draw circles from A and B as centers whose radii are in the ratio m/n. Let these circles intersect at the point P. At P bisect the angle between PA and PB internally and externally by the lines PM and PN respectively. The line AB is then divided at M internally in the ratio $MA/MB = m/n$ and externally at N in the ratio $NA/NB = m/n$, because the bisectors of any angle of a triangle divide the base into segments proportional to the adjacent sides. Since the external and internal bisectors of any angle must be at right angles to each other, PM is perpendicular to PN for any position of P. Hence

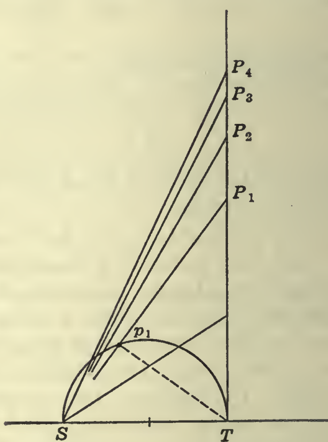


FIG. 156.—Construction of the Constant Products $SP_n \times Sp_n = c^2$.

the locus of P is a circle, since it is the vertex of a right triangle described on the fixed hypotenuse MN.

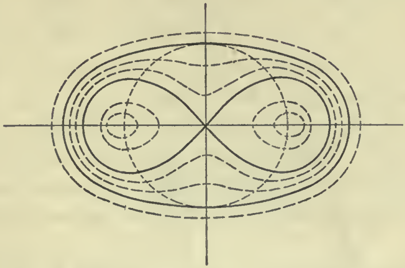


Fig. 157.—The Lemniscate and the Cassinian Ovals.

If e stands for the fixed ratio r_1/r_2 and if $AB = 2a$, the student should show:

$$MA = \frac{2ae}{1+e} \qquad OM = \frac{a(1-e)}{1+e}$$

$$NA = \frac{2ae}{1-e} \qquad MC = \frac{2ae}{1-e^2}$$

$$MN = \frac{4ae}{1-e^2} \qquad OC = \frac{a(1+e^2)}{1-e^2}$$

The equation of the circle should then be found referred to origin O or M.

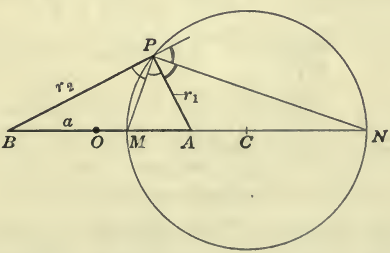


Fig. 158.—Construction of the Curve $r_1/r^2 = m/n$, or the circle MPN.

If a large number of circles be drawn for different values of e, and if similar circles be described about B, then these series of circles are known as the **dipolar circles**. See Fig. 159. In physics it is found that these circles are the equipotential lines about two parallel wires perpendicular to the plane of the paper at A and B and carrying electricity of opposite sign.

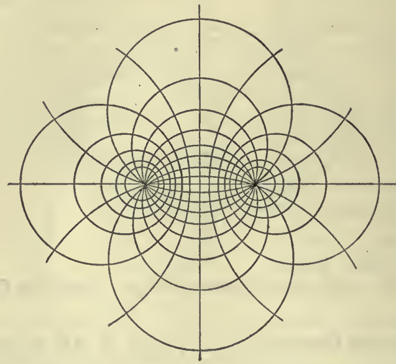


Fig. 159.—The Dipolar Circles, or a Family of Circles made by drawing $r_1/r_2 = e$ for Various Values of e.

Exercises

1. Draw the locus satisfying the condition that the ratio of the distances of any point from two fixed points ten units apart is 2/3.

2. Draw the two circles which divide a line of length 14 internally and externally in the ratio 3/4.

228. The Cycloid. The **cycloid** is the curve traced by a point on the circumference of a circle, called the **generating circle,**

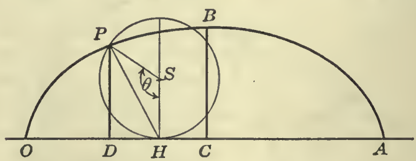


Fig. 160.—Definition of the Cycloid.

which rolls without slipping on a fixed line called the **base.** To find the equation of the cycloid, let OA, Fig. 160, be the base, P the tracing point of the generating circle in any one position, and θ the angle between the radius SP and the line SH to the point of contact with the base. Since P was at O when the circle began to roll,

$$OH = a\theta$$

if a be the radius of the generating circle. Since $x = OD$ and $y = PD$, we have:

$$\mathbf{x} = OH - SP \sin \theta = \mathbf{a}(\theta - \sin \theta) \tag{1}$$

$$\mathbf{y} = HS - SP \cos \theta = \mathbf{a}(1 - \cos \theta) \tag{2}$$

These are the parametric equations of the curve. For most purposes these are more useful than the Cartesian equation. It is readily seen from the definition of the curve, that the locus consists of an unlimited number of loops above the x-axis, with points of contact with the x-axis at intervals of $2\pi a$ (the circumference of the generating circle) and with maximum points at $x = \pi a$, $3\pi a$, etc.

From the second of the parametric equations we may write:

$$1 - \cos \theta = y/a \tag{3}$$

The expression $(1 - \cos \theta)$ is frequently called the *versed sine* of θ, and is abbreviated **vers** θ. Hence we have:

$$\theta = \text{vers}^{-1}\, y/a \tag{4}$$

Also from (3): $\cos \theta = (a - y)/a$

Hence: $$\sin \theta = \sqrt{1 - \cos^2 \theta} = \frac{1}{a}\sqrt{2ay - y^2} \tag{5}$$

whence substituting (4) and (5) in the first of the parametric equations we have:

$$\mathbf{x} = \mathbf{a}\,\mathbf{vers}^{-1}\,(\mathbf{y}/\mathbf{a}) - \sqrt{\mathbf{2ay - y^2}} \tag{6}$$

which is the Cartesian equation of the cycloid, with the origin O at a cusp of the curve.

229. Graphical Construction of the Cycloid. To construct the cycloid, Fig. 161, draw a circle of radius 1.15 inches and divide the circumference into thirty-six equal parts. Draw horizontal lines through each point of division exactly as in the construction of the sinusoid, Fig. 59. Lay off uniform intervals of 1/5 inch each on the x-axis, marked 1, 2, 3, . . . Then from the point of division of the circle p_1 lay off the distance $O1$ to the right. From p_2 lay off $O2$ to the right, from p_3 lay off $O3$ to the right, etc. The points thus determined lie on the cycloid. The number of divisions of the circumference is of course immaterial except that an even number of division is convenient, and except that the divisions laid off on the base OA must be the same length as the arcs laid off on the circle.

Note that by the process of construction above, the vertical distances from OX to points on the curve are proportional to $(1 - \cos \theta)$ and that the horizontal distances from OY to points on the curve are proportional to $(\theta - \sin \theta.)$

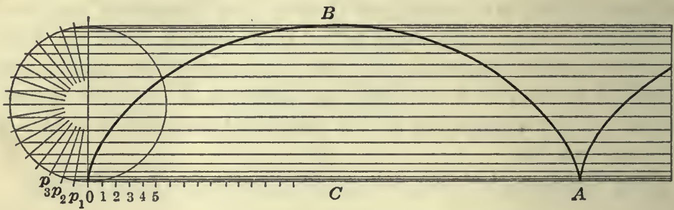


FIG. 161.—Construction of the Cycloid.

The analogy of the cycloid to the sine curve is brought out by Fig. 162. A set of horizontal lines are drawn as before and also a sequence of semicircles spaced at horizontal intervals equal to the intervals of arc on the circle. The plane is thus divided into a large number of small quadrilaterals having two sides straight and two sides curved. Starting at O and sketching the

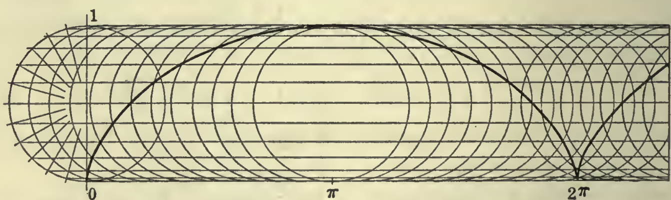


FIG. 162.—Analogy of the Cycloid to the Sinusoid.

diagonals of successive cornering quadrilaterals the cycloid is traced. If, instead of the sequence of circles, uniformly spaced vertical straight lines had been used, the sinusoid would have been drawn. The sinusoid on that account is frequently called the "*companion to the cycloid.*"

230. Epicycloids and Hypocycloids. The curve traced by a point attached to the circumference of a circle which rolls without slipping on the circumference of a fixed circle is called an **epicycloid** or a **hypocycloid** according as the rolling circle touches the outside or inside of the fixed circle. If the tracing point is not on the circumference of the rolling circle but on a radius or radius produced, the curve it describes is called a **trochoid** if the circle rolls upon a straight line, or an **epitrochoid** or a **hypotrochoid** if the circle rolls upon another circle. These curves will be discussed in the calculus.

Exercises

1. Construct a cycloid by dividing a generating circle of radius 1.15 inches into twenty-four equal arcs and dividing the base into intervals 3/10 inch each.

2. Compare the cycloid of length 2π and height 1 with a semi-ellipse of length 2π and height 1.

3. Write the parametric equations of a cycloid for origin C, Fig. 160.

4. Write the parametric equations of a cycloid for origin B, Fig. 160.

5. Find the coördinates of the points of intersections of the cycloid with the horizontal line through the center of the generating circle.

6. Show that the top of a rolling wheel travels through space twice as fast as the hub of the wheel.

7. By experiment or otherwise show that the tangent to the cycloid at any point always passes through the highest point of the generating circle in the instantaneous position of the circle pertaining to that point.

Exercises for Review

1. Simplify the product:

$$(x - 2 - \sqrt{3})(x - 2 - i\sqrt{3})(x - 2 + \sqrt{3})(x - 2 + \sqrt{i3}).$$

2. Express in the form $c \cos (a - b)$ the binomial:

$$30 \cos a + 40 \sin a.$$

3. Find $\tan \theta$ by means of the formula for $\tan (A + B)$, if $\theta = \tan^{-1} 1/2 + \tan^{-1} 1/3$.

4. Find $\sin \theta$ if $\theta = \sin^{-1} 1/5 + \sin^{-1} 1/7$.

5. Find the equation of a circle whose center is the origin and which passes through the point 14, 17.

6. The first of the following tests was made in 1875 with the automatic air brake on a train composed of cars weighing 30,000 pounds. The second in 1907 with the "LN" brake on a train composed of cars weighing 84,000 pounds. Find by use of logarithmic paper the equation connecting the speed and the distance run after application of the brakes.

Distance run after application of brake		Corresponding speed
1875	1907	
0 feet	0 feet	57.3 miles per hour
		56.0 miles "
50 feet	70 feet	55.0 miles "
200 feet	220 feet	50.0 miles "
350 feet	360 feet	45.0 miles "
500 feet	500 feet	40.0 miles "
820 feet	770 feet	25.0 miles "
950 feet	880 feet	15.0 miles "
980 feet	922 feet	10.0 miles "
1,010 feet	940 feet	5.0 miles "
1,020 feet	954 feet	0.0 miles "

7. Discuss the curve:

$$x = a\theta$$
$$y = a(1 - \cos\theta).$$

8. Graph on polar paper:

$$\rho^2 = a^2 \cos 2\theta.$$

9. A fixed point located on one leg of a carpenter's "square" traces a curve as the square is moved, the two arms of the square, however, always passing through two fixed points A and B. Find the equation of the curve.

10. Find the parametric equations of the oval traced by a point attached to the connecting rod of a steam engine.

11. The length of the shadow cast by a tower varies inversely as the tangent of the angle of elevation of the sun. Graph the length of the shadow for various elevations of the sun.

12. From your knowledge of the equations of the straight line and circle, graph:

$$y = ax + \sqrt{a^2 - x^2}.$$

(See Shearing Motion, §37.)

13. In the same manner, sketch:

$$y = a + x + \sqrt{a^2 - x^2}.$$

14. Graph the curve:

$$y = a/x + bx^2.$$

Has this curve a minimum value for all positive values of a and b?

15. Find by use of logarithmic paper the equations of the curves of Fig. 163. These curves give the amounts in cents per kilowatt-hour that must be added to price of electric power to meet fixed charges of certain given annual amounts for various load factors.

16. The angle of elevation of a mountain top seen from a certain point is 29° 4′. The angle of depression of the image of the mountain

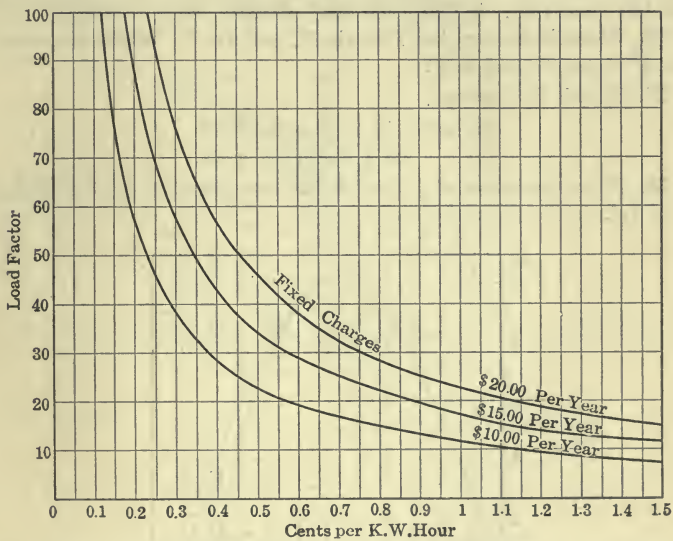


FIG. 163.—Annual Fixed Charges of \$10, \$15 and \$20 of an Hydro-electric Plant, Reduced to Cents per kw-hour for Various Load Factors.

top seen in a lake 230 feet below the observer is 31° 20′. Find the height and horizontal distance of the mountain top, and produce a single formula for the solution of the problem.

17. Find the points of intersection of the curves:

$$x^2 + y^2 = 4$$
$$y^2 = 4x.$$

18. Solve $110x^{-4} + 1 = 21x^{-2}$.

19. Solve $3(x - 7)(x - 1)(x - 2) = (x + 2)(x - 7)(x + 3)$.

20. Solve $\sin x \cos x = 1/4$.

21. State the remainder theorem and illustrate by an example.

22. Find the compound interest on \$1000 for twenty-five years at 5 per cent. *Show how* to solve by means of progressions.

23. The curve $y^2 = x^4$ appears in which quadrants? In what quadrants is $y^4 = x^6$? Compare the curves $x^3y^2 = 1$ and $x^2y^3 = 1$.

24. Which trigonometric functions of θ increase as θ increases in the first quadrant? Which decrease?

25. Given $\sin 30° = 1/2$, $\cos 45° = \sqrt{1/2}$. Find the following: $\sin 150°$, $\cos 135°$, $\sin 225°$, $\cos 300°$, $\sin 330°$, $\sin (-30°)$.

26. Which is greater, $\tan 7°$ or $\sin 7°$, and why? Which is greater, $\sec 5°$ or $\csc 5°$, and why?

27. Sketch the curves:

$$(a)\quad x^2 + 4x + y^2 - 6y = 12$$
$$(b)\quad x^2 + 4y^2 + 6y = 21.$$

28. From the graph of $y = x^2$ obtain the graph of $4y = x^2$ and of $y = 4x^2$.

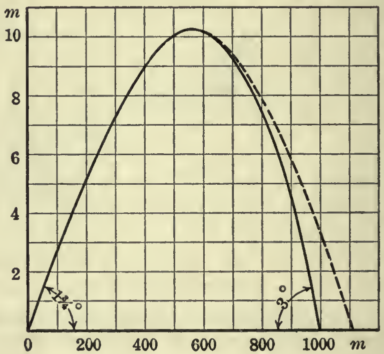


Fig. 164.—Trajectory of a German Army Bullet for a Range of 1000 Meters.

29. Given, $\cos \theta = 2/5$; find $\sin \theta$, $\tan \theta$ and $\cot \theta$.

30. Find the equations of the six straight lines determined by the intersections of:

$$x^2 + y^2 = 25$$
$$x^2 - y^2 = 7.$$

31. In Fig. 164 the full drawn curve is the trajectory of the projectile of a German Army bullet for a range of 1000 meters. The dotted curve is the theoretical trajectory that would have been described by the bullet if there had been no air resistance. The dotted curve is a parabola (of second degree). Find its equation, taking the necessary numerical data from the diagram.

32. Find the maximum value of ρ if $\rho = 3 \cos \theta - 4 \sin \theta$.

33. Find the maximum value of y if $y = \sqrt{3} \cos x - \sin x$, and find the value of x for which y is a maximum.

34. In Fig. 165 let $ABCO$ be a square of side a. Show that for all positions of ON, $CM \times AN = a^2$, and hence show how to use this diagram in the construction of a lemniscate.

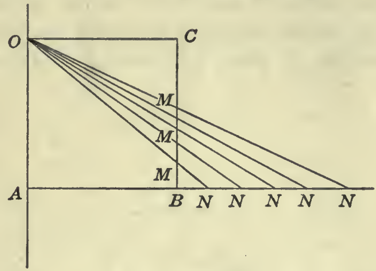


Fig. 165.—Construction of a Constant Product $CM \times AN = \overline{AB}^2$.

CHAPTER XIII

THE CONIC SECTIONS

231. The Focal Radii of the Ellipse. Draw any ellipse with major and minor circles of radii a and b respectively, as in Fig. 166. Draw tangents, II' and KK', to the minor circle at the extremities of the minor axes and complete the rectangle $II'KK'$. The points F_1 and F_2, in which IK and $I'K'$ cut the major axis, are

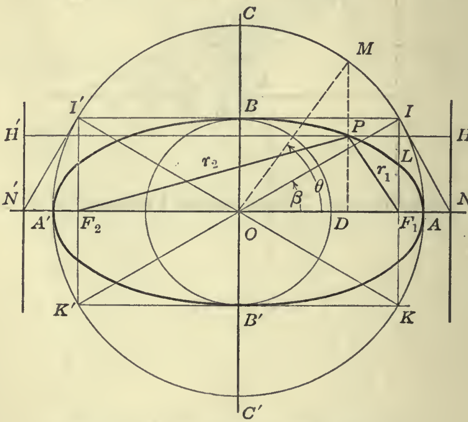


FIG. 166.—Properties of the Ellipse.

called the **foci** of the ellipse. From any point on the ellipse draw the focal radii $PF_1 = r_1$ and $PF_2 = r_2$, as shown in the figure. Represent the distance OF_1 or its equal OF_2 by c. Then it follows from the triangle OIF_1 that:

$$\mathbf{a^2 = b^2 + c^2} \tag{1}$$

This is one of the fundamental relations between the constants of the ellipse.

From the triangles PF_1D and PF_2D there follows:

$$r_1^2 = (c - x)^2 + y^2 \tag{2}$$

$$r_2^2 = (c + x)^2 + y^2 \tag{3}$$

But the equation of the ellipse is

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

or

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \tag{4}$$

Substituting this value of y^2 in (2)

$$r_1^2 = c^2 - 2cx + x^2 + \frac{b^2}{a^2}(a^2 - x^2) \tag{5}$$

$$= c^2 - 2cx + x^2 + b^2 - \frac{b^2}{a^2}x^2$$

or by (1)

$$= a^2 - 2cx + x^2\left[1 - \frac{b^2}{a^2}\right] \tag{6}$$

Substituting

$$1 - \frac{b^2}{a^2} = \frac{a^2 - b^2}{a^2} = \frac{c^2}{a^2}$$

we obtain

$$r_1^2 = a^2 - 2cx + \frac{c^2x^2}{a^2} \tag{7}$$

$$= \left[a - \frac{cx}{a}\right]^2 \tag{8}$$

Therefore:

$$r_1 = a - \frac{cx}{a} \tag{9}$$

Likewise, from (3), by exactly the same substitutions, there follows:

$$r_2 = a + \frac{cx}{a} \tag{10}$$

From (9) and (10) by addition:

$$\mathbf{r_1 + r_2 = 2a} \tag{11}$$

Hence *in any ellipse the sum of the focal radii is constant and equal to the major axis.*

The converse of this theorem, namely, *if the sum of the focal radii of any locus is constant, the curve is an ellipse,* can readily be proved. It is merely necessary to substitute the values of r_1 and r_2 from (2) and (3) in equation (11), and simplify the resulting equation in x and y; or first square (11) and then substitute r_1 and r_2 from (2) and (3). There results *an equation of the second degree lacking the term xy and having the terms containing x^2 and y^2 both* *present and with coefficients of like signs.* By §**77**, such an equation represents an ellipse.

Hence the ellipse might have been defined as the *locus of a point, the sum of the distances of which from two fixed points is constant.*

An ellipse can be drawn by attaching a string of length $2a$ by pins at the points F_1 and F_2 and tracing the curve by a pencil so guided that the string is always kept taut. Or better, take a string of length $2a + 2c$ and form a loop enclosing the two pins; the entire curve can then be drawn with one sweep of the pencil.

The focal radii may also be evaluated in terms of the parametric or eccentric angle θ. The student may regard the following demonstration of the truth of equation (11) as simpler than that given above:

Since $x = a \cos \theta$, and $y = b \sin \theta$

$$r_1^2 = b^2 \sin^2 \theta + (c - a \cos \theta)^2 \qquad (12)$$

$$= b^2 \sin^2 \theta + c^2 - 2ac \cos \theta + a^2 \cos^2 \theta \qquad (13)$$

To put the right side in the form of a perfect square, write $b^2 = a^2 - c^2$. Then:

$$r_1^2 = a^2 \sin^2 \theta - c^2 \sin^2 \theta + c^2 - 2ac \cos \theta + a^2 \cos^2 \theta$$

$$= a^2 - 2ac \cos \theta + c^2 \cos^2 \theta \qquad (14)$$

Whence:

$$\mathbf{r_1 = a - c \cos \theta} \qquad (15)$$

Likewise:

$$\mathbf{r_2 = a + c \cos \theta} \qquad (16)$$

Whence:

$$r_1 + r_2 = 2a$$

232. The Eccentricity. The ratio c/a measures, in terms of a as unit, the distance of either focus from the center of the ellipse. This ratio is called the **eccentricity** of the ellipse. In the triangle IF_1O, the ratio c/a is the cosine of the angle F_1OI, represented in what follows by β. Calling the eccentricity e, we have:

$$\mathbf{e = c/a = \cos \beta} \qquad (1)$$

The ellipse is made from the major circle by contracting its ordinates in the ratio $m = b/a$, or by orthographic projection of the circle through the angle of projection:

$$\alpha = \cos^{-1} b/a$$

Hence, as companion to (1) we may write:

$$m = b/a = \cos\alpha = \sin\beta \tag{2}$$

233. The Ratio Definition of the Ellipse. In Fig. 166, let the tangents to the major circle at I and I' be drawn. Draw a perpendicular to the major axis produced at the points cut by these tangents. These two lines are called the **directrices** of the ellipse.

We shall prove that the ratio PF_1/PH (or PF_2/PH') is constant for all positions of P. From **§231,** equation (9) or (15),

$$r_1 = a - c\cos\theta \tag{1}$$

From the figure,

$$ON = a\sec ION = a\sec\beta \tag{2}$$

But:

$$\cos\beta = c/a$$

Hence:

$$ON = a^2/c \tag{3}$$

But

$$PH = ON - x$$

Therefore

$$PH = a^2/c - a\cos\theta \tag{4}$$

Hence from (1) and (4):

$$\frac{r_1}{PH} = PF_1/PH = \frac{a - c\cos\theta}{a^2/c - a\cos\theta}$$

$$= \frac{c}{a}\,\frac{a - c\cos\theta}{a - c\cos\theta}$$

or

$$\mathbf{PF_1/PH = c/a = e = \cos\beta} \tag{5}$$

A similar proof holds for the other focus and directrix. Thus, for any point on the ellipse the distance to a focus bears a fixed ratio to the distance to the corresponding directrix. From (5), the ratio is seen to be less than unity.

Assuming the converse of the above, the ellipse might have been defined as follows: *The ellipse is the locus of a point whose distance from a fixed point (called the focus) is in a constant ratio less than unity to its distance from a fixed line (called the directrix).*

If, in any ellipse, $c = 0$, it follows that b must equal a and the ellipse reduces to a circle. If c is nearly equal to a, then from the equation:

$$a^2 = b^2 + c^2$$

it follows that the semi-minor axis b must be very small. That is, for an eccentricity nearly unity the ellipse is very slender.

If the sun be regarded as fixed in space, then the orbits of the planets are ellipses, with the sun at one focus. (This is "*Kepler's First Law.*") The eccentricity of the earth's orbit is 0.017. The orbit of Mercury has an eccentricity of about 0.2, which is greater than that of any other planet.

Exercises

Find the eccentricities and the distance from center to foci of the following ellipses:

1. $x^2/9 + y^2/4 = 1$.

2. $y = (2/3)\sqrt{36 - x^2}$.

3. $25x^2 + 4y^2 = 100$.

4. $2y = \sqrt{1 - x^2}$.

5. $9x^2 + 16y^2 = 14$.

6. $2x^2 + 3y^2 = 1$.

Find the equation of the ellipse from the following data:

7. $e = 1/2$, $a = 4$. Draw this ellipse.

8. $c = 4$, $a = 5$.

9. $r_1 = 6 - 2x/3$, $r_2 = 6 + 2x/3$.

10. $r_1 = 5 - 4 \cos \theta$, $r_2 = 5 + 4 \cos \theta$.

Solve the following exercises:

11. Find the eccentricity of the ellipse made by the orthographic projection of the circle $x^2 + y^2 = a^2$ through the angle 60°.

12. The angle of projection of a circle $x^2 + y^2 = a^2$ by which an ellipse is formed is α. Show that the eccentricity of the ellipse is $\sin \alpha$.

13. A circular cylinder of radius 5 is cut by a plane making an angle 30° with the axis. Find the eccentricity of the elliptic section.

14. If the greatest distance of the earth from the sun is 92,500,000 miles, find its least distance. (Eccentricity of earth's orbit = 0.017.)

15. In the ellipse $x^2/25 + y^2/16 = 1$, find the distance between the two directrices.

16. Write the equation of the ellipse whose foci are (2,0), (− 2, 0), and whose directrices are $x = 5$ and $x = -5$.

17. Prove equation 11 §231 by transposing one radical in:

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

squaring, and reducing to an identity.

234. The Latus Rectum. The double ordinate through the focus is called the **latus rectum** of the ellipse. The value of the semi-latus rectum is readily formed from the equation

$$y=(b/a)\sqrt{a^2-x^2}$$

by substituting c for x. If l represents the corresponding value of y,

$$l=(b/a)\sqrt{a^2-c^2}=b^2/a \tag{1}$$

since $a^2-c^2=b^2$. Hence the entire latus rectum is represented by:

$$2l=\frac{2b^2}{a} \tag{2}$$

Equation (1) may also be written:

$$\begin{aligned} l&=b\sqrt{1-c^2/a^2} \\ &=b\sqrt{1-e^2} \end{aligned} \tag{3}$$

In Fig. 166 the distances AF, AN, ON, OB, OF, FN may readily be expressed in terms of a and e as follows in equations (4) to (10). The addition of the formulas (11), (12), (13) brings into a single table all the important formulas of the ellipse.

$$\mathbf{AF_1}=a-c=\mathbf{a(1-e)} \tag{4}$$

$$\mathbf{AN}=\frac{AF_1}{e}=\frac{\mathbf{a(1-e)}}{\mathbf{e}} \tag{5}$$

$$\mathbf{ON}=a\sec\beta=\frac{\mathbf{a}}{\mathbf{e}} \tag{6}$$

$$\mathbf{e}=\mathbf{cos}\,\beta \tag{7}$$

$$\mathbf{OB}=\mathbf{b}=\mathbf{a\,sin}\,\beta=\mathbf{a}\sqrt{\mathbf{1-e^2}} \tag{8}$$

$$\mathbf{OF_1}=\mathbf{c}=\mathbf{ae} \tag{9}$$

$$\mathbf{F_1N}=ON-c=\mathbf{a(1-e^2)/e} \tag{10}$$

$$\mathbf{l}=\mathbf{b^2/a}=\mathbf{a(1-e^2)} \tag{11}$$

$$\mathbf{r_1}=\mathbf{a-ex}=\mathbf{a-x\,cos}\,\beta \tag{12}$$

$$\mathbf{r_2}=\mathbf{a+ex}=\mathbf{a+x\,cos}\,\beta \tag{13}$$

Exercises

1. Find the value in miles of OF for the case of the earth's orbit.

2. Find the value of β for the earth's orbit. (Use the S functions of the logarithmic table.)

3. In the ellipse $y = (2/3)\sqrt{36 - x^2}$ find the length of the latus rectum and the value of e.

4. The eccentricity of an ellipse is 3/5 and the latus rectum is 9 units. Find the equation of the ellipse.

5. In (*a*) $x^2 + 4y^2 = 4$ and (*b*) $2x^2 + 3y^2 = 6$ find the latus rectum, the eccentricity and the distances ON and AF.

6. Determine the eccentricities of the ellipses,

(*a*) $y^2 = 4x - (1/2)x^2$ (*b*) $y^2 = 4x - 2x^2$.

7. Find the equation of an ellipse whose minor axis is 10 units and in which the distance between the foci is 10.

8. Find the equation of an ellipse whose latus rectum is 2 units and minor axis is 2.

9. The distance from the focus to the directrix is 16 units. An ellipse divides the distance between focus and directrix externally and internally in the ratio 3/5. Find the equation of the ellipse.

10. The axes of an ellipse are known. Show how to locate the foci.

11. In an ellipse $a = 25$ feet, $e = 0.96$. What are the values of c and b?

12. For a certain comet (Tempel's) the semi-major axis of the elliptic orbit is 3.5, and $c = 1.4$ on a certain scale. For another comet (Enke's) $a = 2.2$, $e = 0.85$. Sketch the curves, taking 3 cm. or 1 inch as unit of measure.

13. If $l = 7.2$, $e = 0.6$, find c, a, b.

14. An ellipse, with center at the origin and major axis coinciding with the x-axis, passes through the points (10, 5) (6, 13). Find the axes of the ellipse.

235. Focal Radii of the Hyperbola. Construct a hyperbola from auxiliary circles of radii a and b, then the transverse axis of the hyperbola is $2a$ and the conjugate axis is $2b$. Unlike the case of the ellipse, b may be either greater or less than a. As previously explained, the asymptotes are the extensions of the diagonals of the rectangles $BTAO$, $BT'A'O$. From the points I, I′, in which the asymptotes cut the a-circle, draw tangents to the a-circle. The points F_1, F_2 in which the tangents cut the axis of the hyperbola are called the **foci**. See Fig. 167.

The distance OF_1 or OF_2 is represented by the letter c. Then, since the triangles F_1IO and OAT are equal, F_1I must equal b, so that we have the fundamental relation between the constants of the hyperbola:

$$\mathbf{a^2 + b^2 = c^2} \tag{1}$$

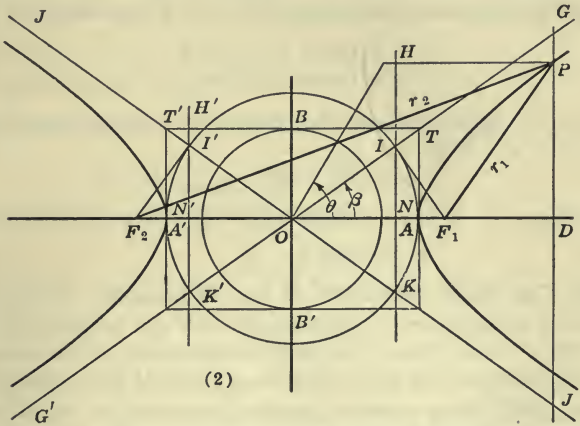


FIG. 167.—Properties of the Hyperbola.

From any point on either branch of the hyperbola draw the focal radii PF_1 and PF_2, represented by r_1 and r_2 respectively. Then from the figure:

$$r_1^2 = (x - c)^2 + y^2 \tag{2}$$

But from the equation of the hyperbola:

$$y^2 = (b^2/a^2)(x^2 - a^2) \tag{3}$$

hence:

$$r_1^2 = (x - c)^2 + b^2(x^2 - a^2)/a^2 \tag{4}$$

$$= (a^2x^2 - 2a^2cx + a^2c^2 + b^2x^2 - a^2b^2)/a^2 \tag{5}$$

$$= (c^2x^2 - 2a^2cx + a^4)/a^2 \tag{6}$$

$$= (cx - a^2)^2/a^2 \tag{7}$$

Hence: $$r_1 = (c/a)x - a \tag{8}$$

In like manner it may be shown that

$$r_2 = (c/a)x + a \tag{9}$$

Hence from (8) and (9) it follows:

$$\mathbf{r_2 - r_1 = 2a} \tag{10}$$

Hence *in any hyperbola, the difference between the distances of any point on it from the foci is constant and equal to the transverse axis.*

The above relation may be derived in terms of the parametric angle θ. Thus, since in any hyperbola $x = a \sec \theta$ and $y = b \tan \theta$,

$$\begin{aligned} r_1^2 &= b^2 \tan^2 \theta + (a \sec \theta - c)^2 \\ &= b^2 \tan^2 \theta + a^2 \sec^2 \theta - 2ac \sec \theta + c^2 \end{aligned}$$

To put the right-hand side in the form of a perfect square, write $b^2 = c^2 - a^2$. Then

$$r_1^2 = c^2 \sec^2 \theta - 2ac \sec \theta + a^2$$

Therefore: $$\mathbf{r_1 = c \sec \theta - a} \tag{11}$$

and: $$\mathbf{r_2 = c \sec \theta + a} \tag{12}$$

236. The Ratio Definition of the Hyperbola. Through the points of intersection of the a-circle with the asymptotes, draw IK, $I'K'$. These lines are called the **directrices** of the hyperbola. It will now be proved that the ratio of the distance of any point of the hyperbola from a focus to its distance from the corresponding directrix is constant. Adopt the notation:

$$c/a = \sec \beta = e \tag{1}$$

Then from the figure:

$$PF_1/PH = r_1/(x - ON) = r_1/(a \sec \theta - a \cos \beta) \tag{2}$$

Substituting r_1 from (11) above:

$$PF_1/PH = (c \sec \theta - a)/(a \sec \theta - a \cos \beta) \tag{3}$$

$$= (a \sec \beta \sec \theta - a)/(a \sec \theta - a \cos \beta) \tag{4}$$

$$= \frac{\sec \beta \sec \theta - 1}{\sec \beta - \cos \theta} = \sec \beta = e = c/a \tag{5}$$

which proves the theorem. The constant ratio e is called the **eccentricity** of the hyperbola, and, as shown by (5), is always greater than unity.

Assuming the converse of the above, it is obvious that the hyperbola might have been defined as follows: *The hyperbola is the locus of a point whose distance from a fixed point (called the focus) is in a constant ratio greater than unity to its distance from a fixed line (called the directrix).*

237. The Latus Rectum. The double ordinate through the focus is called the **latus rectum** of the hyperbola. The value of the semi-latus rectum is readily found from the equation:

$$y = (b/a)\sqrt{x^2 - a^2}$$

by substituting c for x. If l represents the corresponding value of y:

$$l = (b/a)\sqrt{c^2 - a^2} = b^2/a \tag{1}$$

Hence the entire latus rectum is represented by:

$$2l = 2b^2/a \tag{2}$$

Equation (1) may also be written:

$$l = b\sqrt{\frac{c^2}{a^2} - 1}$$

$$l = b\sqrt{e^2 - 1} \tag{3}$$

In Fig. 167 the distances AF_1, AN, ON, OB, OF_1, F_1N may readily be expressed in terms of a and e, as follows in equations (4) to (8). Collecting in a single table the other important formulas for the hyperbola, we have:

$$AF_1 = c - a = a(e - 1) \tag{4}$$

$$AN = AF_1/e = a(e - 1)/e \tag{5}$$

$$ON = a\cos\beta = a/e \tag{6}$$

$$e = \sec\beta \tag{7}$$

$$OB = b = a\tan\beta = a\sqrt{e^2 - 1} \tag{8}$$

$$OF_1 = c = ae$$

$$F_1N = c - ON = ae - a/e = a(e^2 - 1)/e \tag{9}$$

$$l = b^2/a = b\sqrt{e^2 - 1} = a(e^2 - 1) \tag{10}$$

$$r_1 = ex - a = x\sec\beta - a \tag{11}$$

$$r_2 = ex + a = x\sec\beta + a \tag{12}$$

The important properties of the hyperbola are quite similar to those of the ellipse. It is a good plan to compare them in parallel columns.

Ellipse	Hyperbola
1. Definition of Foci and Focal Radii	1. Definition of Foci and Focal Radii
2. $a^2 = b^2 + c^2$	2. $a^2 + b^2 = c^2$
3. $r_1 + r_2 = 2a$	3. $r_2 - r_1 = 2a$
4. Eccentricity, $e = \frac{c}{a} = \cos\beta$	4. Eccentricity, $e = \frac{c}{a} = \sec\beta$
5. Definition of Directrices	5. Definition of Directrices
6. The Ratio Property, $\frac{PF_1}{PH} = e$	6. The Ratio Property, $\frac{PF_1}{PH} = e$
7. The Latus Rectum $= \frac{2b^2}{a}$	7. The Latus Rectum $= \frac{2b^2}{a}$

Exercises

1. Find the eccentricity and axes of $x^2/4 - y^2/16 = 1$.

2. Find the eccentricity and latus rectum of the hyperbola conjugate to the hyperbola of the preceeding exercise.

3. A hyperbola has a transverse axis equal to 14 units and its asymptotes make an angle of 30° with he x-axis. Find the equation of the hyperbola.

4. Find the latus rectum and locate the foci and asymptotes of $4x^2 - 36y^2 = 144$.

5. Locate the directrices of the hyperbola of the preceding exercise.

6. In Fig. 167 show that $r_2 = GK'$ and $r_1 = GI$ and hence that $r_2 - r_1 = IK'$ or $2a$.

7. Find the equation of the hyperbola having latus rectum 4/3 and $a = 2b$.

8. The eccentricity of a hyperbola is 3/2 and its directrices are the lines $x = 2$ and $x = -2$. Write the equation and draw the curve with its asymptotes, a-circle, b-circle, and foci.

9. Find the eccentricity and axes of $3x^2 - 5y^2 = -45$.

10. Find the eccentricity of the rectangular hyperbola.

11. Describe the shape of a hyperbola whose eccentricity is nearly unity. Describe the form of a hyperbola if the eccentricity is very large.

12. Describe the hyperbola if $b/a = 2$, but a very small.

13. Write the equation of the hyperbola if (1) $c = 5$, $a = 3$; (2) $c = 25$, $a = 24$; (3) $c = 17$, $b = 8$.

14. Describe the locus:

$$(x + 1)^2/7 - (y - 3)^2/5 = 1.$$

15. Find the equation of the hyperbola whose center is at the origin and whose transverse axis coincides with the x-axis and which passes through the points $(4.5, -1)$, $(6, 8)$.

238. The Polar Equation of the Ellipse and Hyperbola. In mechanics and astronomy the polar equations of the ellipse and hyperbola are often required with the pole or origin at the right focus in the case of the ellipse and at the left focus in the case of the hyperbola. In these positions the radius vector of any point on the

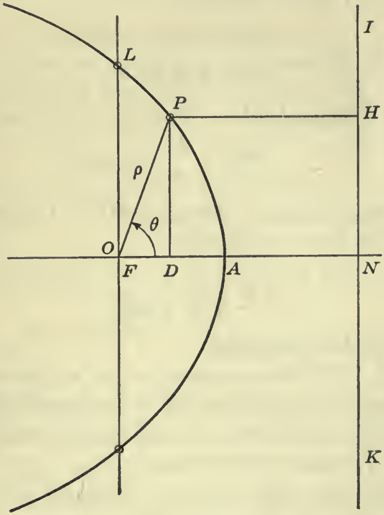


Fig. 168.—Polar Equation of a Conic.

curve will increase with the vectorial angle when $\theta < 180°$. To obtain the polar equation of the ellipse and hyperbola, make use of the ratio property of the curves, namely: that the locus of a point whose distances from a fixed point (called the focus) is in a constant ratio e to its distances from a fixed line (called the directrix), is an ellipse if $e < 1$ or a hyperbola if $e > 1$. In Fig. 168 let F be the fixed point or focus, IK the fixed line or directrix, P the moving point, and $FL = l$ the semi-latus rectum. Then the problem is to find the polar equation from the equation

$$\frac{PF}{PH} = e \tag{1}$$

If e is left unrestricted in value, the work and the result will apply equally well either to the ellipse or to the hyperbola.

When the point P occupies the position L, Fig. 168, we have $PF = l$ and $PH = FN$, whence from (1)

$$FN = \frac{l}{e} \tag{2}$$

Take the origin of polar coördinates at F, and also take $FP = \rho$ and the angle $AFP = \theta$. Then:

$$PH = FN - FD \tag{3}$$

$$FD = \rho \cos \theta \tag{4}$$

Hence from (2), (3) and (4)

$$PH = \frac{l}{e} - \rho \cos \theta \tag{5}$$

Substituting these values of FP and PH in (1), clearing of fractions and solving for ρ, we obtain

$$\rho = \frac{1}{1 + e \cos \theta} \tag{6}$$

which is the equation required.

When $e < 1$, (6) is the equation of an ellipse with pole at the right-hand focus. When $e > 1$, (6) is the equation of a hyperbola with the pole at the left focus; in both cases the origin has been so selected that ρ increases as θ increases.

NOTE: Calling FN (Fig. 168) $= n$, equation (1) above may be written in rectangular coördinates:

$$\frac{\sqrt{x^2 + y^2}}{n - x} = e \tag{7}$$

or,

$$x^2 + y^2 = e^2(n - x)^2 \tag{8}$$

which may be reduced to the form:

$$\left(x + \frac{ne^2}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2n^2}{(1 - e^2)^2} \tag{9}$$

By §§**77** and **87** this represents an ellipse if $e < 1$ or a hyperbola if $e > 1$. Thus starting with the ratio definition (7) we have proved that the curve is an ellipse or a hyperbola; that is, we have proved the statements in italics at bottom of pp. 401 and 406.

Exercises

1. Graph on polar paper, form $M3$, the curve $\rho = \dfrac{6}{1 + e \cos \theta}$ for $e = 2$; also for $e = 1/2$, also for $e = 1$.

It will be sufficient in graphing to use $\theta = 0°, 30°, 60°, 90°, 120°, 150°, 180°, 210°, \ldots, 360°$.

2. Write the polar equation of an ellipse whose semi-latus rectum is 6 feet and whose eccentricity is 1/3.

3. Write the polar equation of an ellipse whose semi-axes are 5 and 3.

4. Discuss equation (6) for the case $e = 0$.

5. Write the polar equation of a hyperbola if the eccentricity be $\sqrt{2}$ and the distance from focus to vertex be 4.

6. Write the polar equations of the asymptotes of

$$\rho = \frac{6}{4 + 5 \cos \theta}.$$

See §87.

7. Compare the curves $\rho = \dfrac{9}{4 + 5 \cos \theta}$ and $\rho = \dfrac{9}{4 - 5 \cos \theta}$.

8. Discuss the equation $\rho = \dfrac{l}{1 + e \cos (\theta - \alpha)}$, in which α is a constant.

239. Ratio Definition of the Parabola. Among the curves of the parabolic type previously discussed, the one whose equation is of the second degree is of paramount importance. On that account when the term **parabola** is used without qualification, it is understood that the curve is the parabola of the second order, whose equation may be written, $y^2 = ax$ or $x^2 = ay$.

The locus of a point whose distance from a fixed point is always equal to its distance from a fixed line is a parabola. In Fig. 169, let F be the fixed point and HK the fixed line. Take the origin at A half way between F and HK. Let P be any point satisfying the condition $PF = PH$. Call $OD = x, PD = y$, and represent the given distance FK by $2p$. Then, from the right triangle PFD:

$$\begin{aligned} PF^2 &= y^2 + FD^2 \qquad (1) \\ &= y^2 + (x - OF)^2 \\ &= y^2 + (x - p)^2 \end{aligned}$$

Since PF by definition equals PH or $x + p$, we have:

$$(x + p)^2 = y^2 + (x - p)^2 \tag{2}$$

whence:

$$\mathbf{y^2 = 4px} \tag{3}$$

which is the equation of the parabola in terms of the focal distance, OF or p.

The double ordinate through F is called the **latus rectum.**

The semi-latus rectum can be computed at once from (3) by placing $x = p$, whence:

$$\mathbf{l = 2p} \tag{4}$$

where l is the semi-latus rectum. Hence the entire latus rectum is $4p$, or the coefficient of x in equation (3).

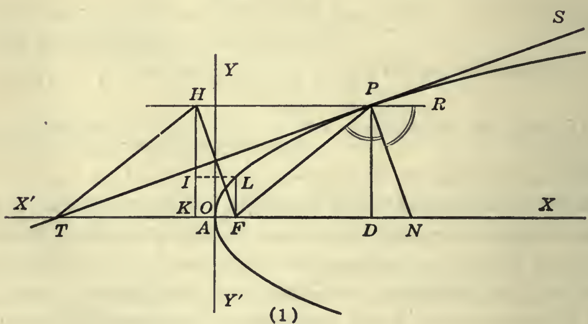


FIG. 169.—Properties of the Parabola $y^2 = 4px$.

In Fig. 169, the quadrilateral $FLIK$ is a square since FL and FK are each equal to $2p$.

240. Polar Equation of the Parabola. In accordance with the ratio definition of the parabola, its polar equation is found at once from equation (6), §**238,** by putting $e = 1$. Hence the polar equation of the parabola is

$$\rho = \frac{l}{1 + \cos\theta} \tag{1}$$

For this equation we may make the following table of values:

θ	ρ
$0°$	$l/2$
$90°$	l
$180°$	∞
$270°$	l

This shows that the parabola has the position shown in Fig. 168. This is the form in which the polar equation of the parabola is used in mechanics and astronomy.

241. The Conics. It is now obvious that a single definition can be given that will include the ellipse, hyperbola and parabola. These curves taken together are called **the conics.** The definition may be worded: *A conic is the locus of a point whose distances from a fixed point* (*called the focus*) *and a fixed line* (*called the directrix*) *are in a constant ratio.* The unity between the three curves was shown by their equation in polar coördinates. Moving the ellipse so that its left vertex passes through the origin, as in **§76,** and writing the hyperbola with the origin at the right vertex (so that both curves pass through the origin in a comparable manner), we may compare each with the parabola as follows:

The ellipse: $$y^2 = 2lx - (b^2/a^2)x^2 \tag{1}$$

The parabola: $$y^2 = 2lx \tag{2}$$

The hyperbola: $$y^2 = 2lx + (b^2/a^2)x^2 \tag{3}$$

In these equations l stands for the semi-latus rectum of each of the curves. These equations may also be written:

$$y^2 = 2lx - (l/a)x^2 \tag{4}$$

$$y^2 = 2lx \tag{5}$$

$$y^2 = 2lx + (l/a)x^2 \tag{6}$$

whence it is seen that if l be kept constant while a be increased without limit, the ellipse and hyperbola each approach the parabola as near as we please. Only for large values of x, if a be large, is there a material difference in the shapes of the curves.

Exercises

1. Write the equation of the circle in the form (1) above.

2. Write the equation of the equilateral hyperbola in the form (3) above.

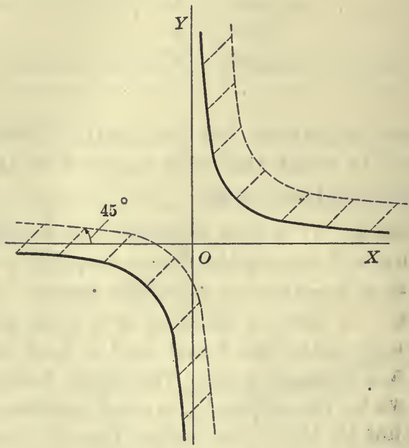


FIG. 170.—A Hyperbola Translated at an Angle of 45° to OX.

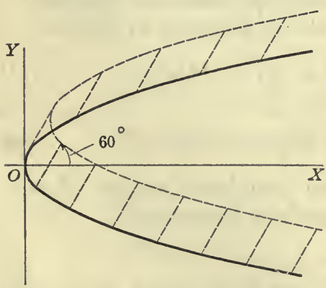


FIG. 171.—A Parabola Translated at an Angle of 60° to OX.

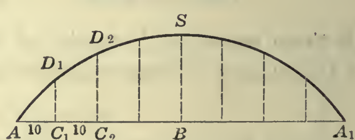


FIG. 172.—Bridge Truss in Form of Circular Segment.

3. Describe the curve:

$$\rho = \frac{l}{1 + \cos(\theta - \alpha)}$$

where α is a constant.

4. In Fig. 170 translate the curve $xy = 1$ by suitable change in the equation to the position shown by the dotted curve, if the translation of each point is unity.

5. In Fig. 171 translate the curve $y^2 = 4px$ by suitable change in the equation to the position shown by the dotted curve, if the distance each point is moved be $3p$.

6. A bridge truss has the form of a circular segment, as shown in Fig. 172. If the total span be 80 yards and the altitude BS be 20 yards, find the ordinates C_1D_1, C_2D_2 erected at uniform intervals of 10 yards along the chord AA_1.

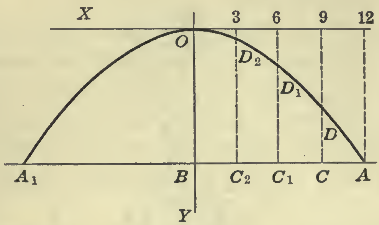


FIG. 173.—Bridge Truss in the Form of a Parabolic Segment.

7. A bridge truss has the form of a parabolic segment, as shown in Fig. 173. The span AA_1 is 24 yards and the altitude OB is 10 yards. Find the length of the ordinates DC, D_1C_1, . . . erected at uniform intervals of 3 yards along the line AA_1.

242.* The Conics are Conic Sections. The curves now known as the conics were originally studied by the Greek geometers as the sections of a circular cone cut by a plane. At first these sections were made by passing a plane perpendicular to one element of a right circular cone. If the angle at the apex of the cone was a right angle, the section was called *the section of the right angled cone*. If the angle at the apex of the cone was less than 90°, the section made by the cutting plane was called the *section of the acute angled cone*. Likewise a third curve was named the *section of the obtuse angled cone*. Thus the curves of three different types now called the parabola, ellipse, and hyperbola were studied. The present names were not introduced until much later, and until it was shown that the three classes of curves could be made respectively by cutting any cone: (1) by a plane parallel to an element; (2) by a plane cutting opposite elements of the same nappe of the cone; (3) by a plane cutting both nappes of the cone. The two nappes of a conical surface, it will be remembered, are the two portions of the surface separated by the apex.

In Fig. 174, let the plane $NDN'D'$, called the cutting plane, cut the lower nappe of a right circular cone in the curve VPV'. We shall prove that this curve is an ellipse.

Let the plane VAV' pass through the axis of the cone. It is then possible to fit into the cone two spheres which will be tangent to the elements of the cone and also tangent to the cutting plane.

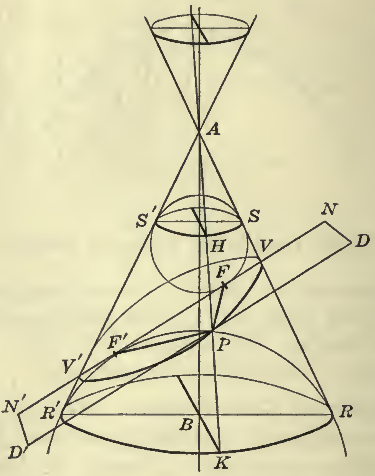


Fig. 174.—Section of a Circular Cone.

For it is merely necessary to locate by plane geometry the circle inscribed in the triangle AVV', and the escribed circle $RF'R'$, and then to rotate these circles about the axis AB to describe the required spheres while the line AR describes the conical surface.

Let the points at which the cutting plane touches the two spheres be called F and F'.

From any point P on the curve VPV' draw lines PF and PF' to the points F and F'. These lines are tangent to the spheres, since each lies in a tangent plane and passes through the point of tangency. Through P draw an element of the cone $AHPK$. The lines PH and PK are also tangents to the upper and lower spheres respectively. Since all tangents to the same sphere from the same external point are equal:

$$PF = PH$$
$$PF' = PK$$

Hence:

$$PF + PF' = PH + PK$$

But $PH + PK$ is an element of the frustum $SHS'\,RKR'$, and hence preserves the same value for all positions of P. Hence:

$$PF + PF' = \text{a constant sum}$$

Therefore the section is an ellipse with foci F and F'.

Let the upper and lower circle of tangency of the spheres and conical surface, namely SHS' and RKR', be produced until they cut the cutting plane in the straight lines ND and $N'D'$. DPD' is a perpendicular at P to the parallel lines ND and $N'D'$. We shall show that the parallel lines ND and $N'D'$ are the directrices of the ellipse.

Since:

$$PF = PH$$

we have

$$PF/PD = PH/PD$$

The two intersecting lines DD' and HK are cut by the parallel planes DNS and $D'N'R'$. Hence we have the proportion:

$$PH/PD = PK/PD' = HK/DD'$$

This last ratio, however, has the same value for all positions of P, since HK is an element of the frustum and since DD' is the fixed distance between the parallel lines ND and $N'D'$.

Therefore with respect to the points F and F' and the lines ND and $N'D'$ the ratio definition of the ellipse applies to the curve VPV'. It is easy to show that the ratio HK/DD is less than unity.

If the cutting plane be passed parallel to the element AR', it is easy to prove that the curve of the section satisfies the ratio definition of the parabola. In case the cutting plane cuts both nappes, one of the tangent spheres lies above the apex and it is easy to show that $PK - PH$ is constant.

243. Tangent to the Parabola. Let us investigate the condition that the line $y = mx + b$ shall be tangent to the parabola $y^2 = 4px$. First find the points of intersection of these loci by solving the two equations for x and y:

$$y = mx + b \tag{1}$$

$$y^2 = 4px \tag{2}$$

as simultaneous equations.

Eliminating y by substituting the value of y from (1) in (2)

$$m^2x^2 + 2mbx + b^2 - 4px = 0 \tag{3}$$

or

$$m^2x^2 + 2(mb - 2p)x + b^2 = 0 \tag{4}$$

Solving for x (see formula for quadratic, Appendix.

$$x = -\frac{mb - 2p}{m^2} \pm \frac{2\sqrt{p^2 - pmb}}{m^2} \tag{5}$$

Therefore there are in general *two* values of x or two points of intersection of the straight line and the parabola. By the definition of a tangent to a curve (§ **146**) the line becomes a tangent to the parabola when the two points of intersection become a single point; that is, when the radical in (5) vanishes. This condition requires that:

$$p^2 - pmb = 0$$

or:

$$b = p/m \tag{6}$$

Therefore when b of equation (1) has this value, the line touches the parabola at but a single point, or is tangent to it. The equation of the tangent is therefore:

$$\mathbf{y = mx + p/m} \tag{7}$$

This line is tangent to the parabola $y^2 = 4px$ for all values of m. Substituting in (5) the value of $b = p/m$, we may find the abscissa of the point of tangency:

$$\mathbf{x_1 = p/m^2} \tag{8}$$

Substituting this value of x in (7) the corresponding ordinate of this point is found to be:

$$\mathbf{y_1 = 2p/m} \tag{9}$$

244. Properties of the Parabola. In Fig. 169, F is the focus, HK is the directrix, PT is a tangent at any point P. The perpendicular PN to the tangent at the point of tangency is called the **normal** to the parabola. The projection DT of the tangent PT on the x-axis is called the **subtangent** and the projection DN of the normal PN on the x-axis is called the **subnormal.** The line through any point parallel to the axis, as PR, is known as a **diameter** of the parabola.

(*a*) *The subtangent to the parabola at any point is bisected by the vertex.* It is to be proved that $OT = OD$ for all positions of P. Now OD is the abscissa of P, which has been found to be p/m^2. From the equation of the tangent:

$$y = mx + p/m$$

the intercept OT on the x-axis is found by putting $y = 0$ and solving for x. This yields:

$$x = -p/m^2$$

This is numerically the same as OD, hence the vertex O bisects DT.

(*b*) *The subnormal to the parabola at any point is constant and equal to the semi-latus rectum.*

The angle DPN has its sides mutually perpendicular to the sides of the angle DTP, hence the angles are equal. Since the tangent of the angle $DTP = m$, therefore:

$$\text{tangent } DPN = m$$

From properties of the right triangle PDN:

$$\begin{aligned} DN &= PD \text{ tangent } DPN \\ &= PD\, m \\ &= (2p/m)m = 2p \end{aligned}$$

Since KF also equals $2p$, we have

$$KF = DN$$

(*c*) *PFTH is a rhombus.* By hypothesis $PF = PH$. To prove the figure $PFTH$ a rhombus it is merely necessary to show that $FT = PH$.

Now:

$$\begin{aligned} FT &= FO + OT \\ PH &= DK = DO + OK \end{aligned}$$

But:

$$OD = OT \text{ and } OK = FO$$

therefore:

$$FT = PH$$

and the figure is a rhombus.

It follows that the two diagonals of the rhombus intersect at right angles on the y-axis.

(*d*) *The normal to a parabola bisects the angle between the focal radius and the diameter at the point.* We are to show that:

$$\angle NPF = \angle NPR$$

Since $FPHT$ is a rhombus:

$$\angle FPT = \angle TPH$$

But:

$$\angle TPH = \angle RPS$$

being vertical angles. From the two right angles NPT and NPS subtract the equal angles last named. There results:

$$\angle \mathbf{FPN} = \angle \mathbf{NPR}$$

It is because of this property of the parabola that the reflectors of locomotive or automobile headlights are made parabolic. The rays from a source of light at F are reflected in lines parallel to the axis, so that, in the theoretical case, a beam of light is sent out in parallel lines, or in a beam of undiminishing strength.

245. To Draw a Parabolic Arc. One of the best ways of describing a parabolic arc is by drawing a large number of tangent lines by the principle of §**244** (*c*). Since in Fig. 169 the tangent is for all positions perpendicular to the focal line FH at the point where the latter crosses OY, it is merely necessary to draw a large number of focal lines, as in Fig. 175, and erect perpendiculars to them at the points where they cross the y-axis.

The equations of the tangent lines in Fig. 175 are of the form:

$$y = mx + p/m \tag{1}$$

in which p is the constant given by the equation of the parabola, and in which m takes on in succession a sequence of values appropriate to the large number of tangent lines of the figure. These lines are said to constitute a **family of lines** and are said to **envelop** the curve to which they are tangent. The curve itself is called the **envelope** of the family of lines.

The curve of the supporting surface of an aeroplane as well as

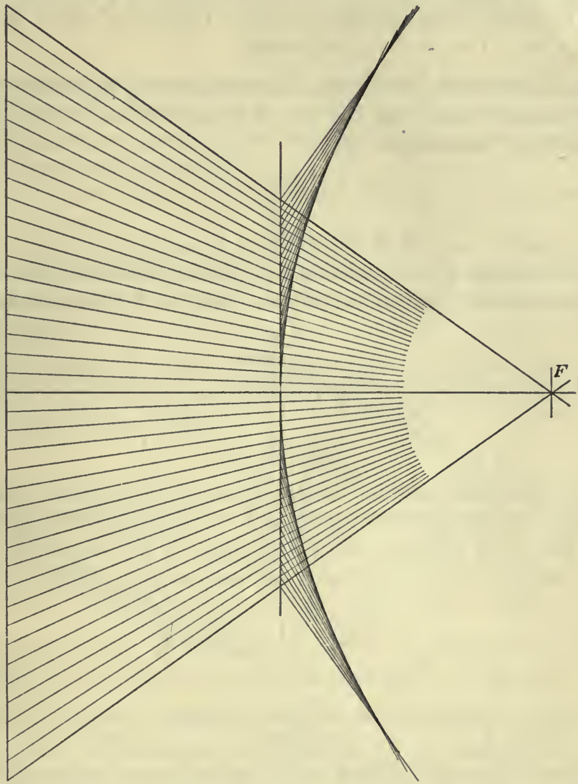


FIG. 175.—Graphical Construction of a Parabolic Arc "by Tangents."

the curve of the propeller blades is a parabolic arc. The curve of the cables of a suspension bridge is also parabolic.

Exercises

1. Write the equation of the parabola which the family $y = mx + 7/2m$ envelops.

2. Draw an arc of a parabola if $p = 3$ inches.

3. At what point is $y = mx + 3/m$ tangent to the parabola $y^2 = 12x$?

4. At what point is $y = mx + 11/m$ tangent to $y^2 = 44x$?

5. Draw the family of lines $y = mx + 1/m$ for $m = 0.4$, $m = 0.6$, $m = 0.8$, $m = 1$, $m = 2$, $m = 4$, $m = 8$.

246. Tangent to the Circle. An equation of a tangent line to a circle can be found as in the case of the parabola above by finding the points of intersection of:

$$y = mx + b \tag{1}$$

and

$$x^2 + y^2 = a^2 \tag{2}$$

and then imposing the condition that the two points of intersection shall become a single point. The value of b that satisfies this

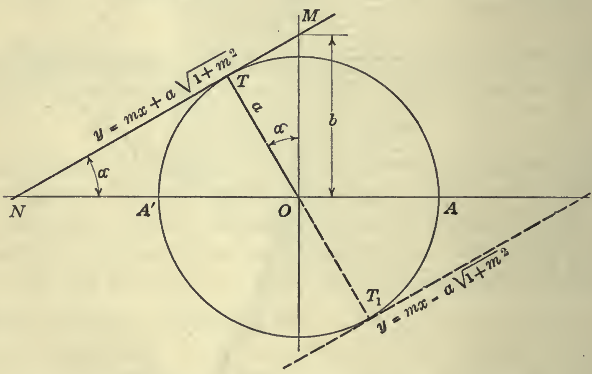


Fig. 176.—The Equation of a Line of Given Slope, Tangent to a Given Circle.

condition when substituted in (1) gives the equation of the required tangent. It is easier to obtain this result, however, by the following method. In Fig. 176 let the straight line be drawn tangent to the circle at T. Let the slope of this line be m. Then $m = \tan ONT = \tan \alpha$, if α be the direction angle of the tangent line. The intercept of the line on the y-axis can be expressed in terms of a and α:

$$b = a \sec \alpha = a\sqrt{1 + m^2} \tag{3}$$

Hence the equation of the tangent to the circle is:

$$\mathbf{y = mx \pm a\sqrt{1 + m^2}}$$

The double sign is written in order to include in a single equation the two tangents of given slope m, as illustrated in the diagram.

Exercises

1. Find the equations of the tangents to $x^2 + y^2 = 16$ making an angle of 60° with the x-axis.

2. Find the equations of the tangents to $x^2 + y^2 = 25$ making an angle of 45° with the x-axis.

3. Find the equation of tangents to $x^2 + y^2 = 25$ parallel to $y = 3x - 2$.

4. Find the equation of tangents to $x^2 + y^2 = 16$ perpendicular to $y = (1/2)x + 3$.

5. Find the equations of the tangents to $(x - 3)^2 + (y - 4)^2 = 25$ whose slope is 3.

247. Normal Equation of Straight Line. The normal equation of the straight line was obtained in polar coördinates in **§69.** The equation was written:

$$\rho \cos (\theta - \alpha) = a \tag{1}$$

In this equation (ρ, θ) are the polar coördinates of any point on the line, a is the distance of the line from the origin and α is the direction angle of a perpendicular to the line from the origin. (See Fig. 177.) Expanding $\cos (\theta - \alpha)$ in (1) we obtain:

$$\rho \cos \theta \cos \alpha + \rho \sin \theta \sin \alpha = a \tag{2}$$

But for any value of ρ and θ, $\rho \cos \theta = x$ and $\rho \sin \theta = y$.

Hence (2) may be written in rectangular coördinates:

$$\mathbf{x \cos \alpha + y \sin \alpha = a} \tag{3}$$

This also is called the **normal equation** of the straight line.

If an equation of any line be given in the form:

$$ax + by = c \tag{4}$$

it can readily be reduced to the normal form. For dividing this equation through by $\sqrt{a^2 + b^2}$:

$$\frac{\mathbf{a}}{\sqrt{\mathbf{a^2 + b^2}}}\,\mathbf{x} + \frac{\mathbf{b}}{\sqrt{\mathbf{a^2 + b^2}}}\,\mathbf{y} = \frac{\mathbf{c}}{\sqrt{\mathbf{a^2 + b^2}}} \tag{5}$$

Now $a / \sqrt{a^2 + b^2}$ and $b / \sqrt{a^2 + b^2}$ may be regarded as the cosine and sine, respectively, of an angle, for a and b are divided by a number which may be represented by the hypotenuse of a right triangle of which a and b are legs. Calling this angle α, equation (5) may be written:

$$x \cos \alpha + y \sin \alpha = d \tag{6}$$

which is of the form (3) above. Inasmuch as the right side of the equation in the normal form represents the distance of the line from the origin, it is best to keep the right side of the equation positive. The value of α and the quadrant in which it lies is then determined by the signs of $\cos \alpha$ and $\sin \alpha$ on the left side of the equation. The angle α may have any value from 0° to 360°.

ILLUSTRATIONS:

(1) Put the equation $3x - 4y = 10$ in the normal form. Here $a^2 + b^2 = 25$. Dividing by 5 we obtain:

$$(3/5)x - (4/5)y = 2$$

The distance of this line from the origin is 2. The angle α is the angle whose cosine is 3/5 and whose sine is $-4/5$. Therefore from the tables:

$$\alpha = 306° \; 52'$$

(2) Put the equation $-3x + 4y = 20$ in the normal form.

Here $\cos \alpha = -3/5$, $\sin \alpha = 4/5$, $a = 4$. Hence $\alpha = 126° \; 52'$.

(3) What is the distance between the lines (1) and (2)? The lines are parallel and on opposite sides of the origin. Their distance apart is therefore $2 + 4$ or 6.

Exercises

1. The shortest distance from the origin to a line is 5 and the direction angle of the perpendicular from the origin to the line is 30°. Write the equation of the line.

2. The perpendicular from the origin upon a straight line makes an angle of 135° with OX, and its length is $2\sqrt{2}$. Find the equation of the line.

3. Write the equation of a straight line in the normal form if $\alpha = 60°$ and $a = \sqrt{3}$.

248. To Translate Any Point a Given Distance in a Given Direction. To move any point the distance d to the right we substitute $(x_1 - d)$ for x. To move the point the distance d in the

y direction we substitute $(y_1 - d)$ for y. To move any point the distance d in the direction α we substitute:

$$\begin{aligned} x &= x_1 - d \cos \alpha \\ y &= y_1 - d \sin \alpha \end{aligned} \qquad (1)$$

which must give the desired position of the new point. It is not necessary to use the subscript attached to the new coördinates if the distinction between the new and old coördinates can be kept in mind without this device.

The circle $x^2 + y^2 = a^2$ moved the distance d in the direction α becomes:

$$(x - d \cos \alpha)^2 + (y - d \sin \alpha)^2 = a^2$$

which may be simplified to:

$$x^2 - 2dx \cos \alpha + y^2 - 2dy \sin \alpha = a^2 - d^2$$

249. Distance of Any Point From Any Line. Let the equation of the line be represented in the normal form:

$$x \cos \alpha + y \sin \alpha = a \qquad (1)$$

and let (x_1, y_1) be any point P in the plane. (See Fig. 177.) If the point (x_1, y_1) is on the same side of the line as the origin, the point can be moved to the line by translating the point the proper distance in the α direction. Let the unknown amount of the required translation be represented by d. To translate the point P the amount d in the α direction, we must substitute for x_1 and y_1 the values:

$$\left.\begin{aligned} x_1 &= x - d \cos \alpha \\ y_1 &= y - d \sin \alpha \end{aligned}\right\} \qquad (2)$$

By hypothesis the point now lies on the line, and therefore the *new* coördinates (x, y) of the point must satisfy the equation of the line. Hence, solving (2) for x and y and substituting their values in (1) we have:

$$(x_1 + d \cos \alpha) \cos \alpha + (y_1 + d \sin \alpha) \sin \alpha = a \qquad (3)$$

Performing the multiplications and solving for the unknown number d, we have:

$$\mathbf{d = -(x_1 \cos \alpha + y_1 \sin \alpha - a)} \qquad (4)$$

This is the distance of (x_1, y_1) from the line. Since this distance would ordinarily be looked upon as a signless or arithmetical number, the algebraic sign may be ignored, and only the absolute value of the expression be used. The negative sign means that the given point lies on the origin side of the line.

Equation (4) may be interpreted as follows:

To find the distance of any point from a given line, put the equation of the line in the normal form, transpose all terms to the left

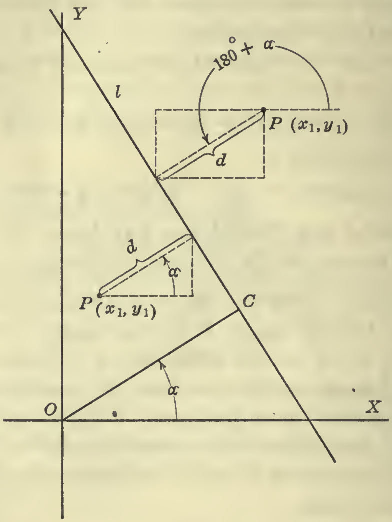


Fig. 177.—Normal Equation of a Line, and the Distance of Any Point from a Given Line.

member and substitute the coördinates of the given point for x and y. The absolute value of the left member is the distance of P from the line.

If the given point P and the origin of coördinates lie on opposite sides of the given line, then the point P (Fig. 177) must be translated in the direction $(180° + \alpha)$ to reach the line. Hence the substitutions are

$$x_1 = x - d \cos (180° + \alpha)$$
$$y_1 = y - d \sin (180° + \alpha)$$

or,

$$x_1 = x + d \cos \alpha$$
$$y_1 = y + d \sin \alpha$$

Solving these for x and y, substituting in the equation of the line, and solving for d we obtain:

$$\mathbf{d = x_1 \cos \alpha + y_1 \sin \alpha - a} \tag{6}$$

The absolute value is of the same form as before. Hence only the one formula (4) is required. When the result in any problem comes out negative it merely means the given point lies on the origin side of the line.

The above facts may be stated in an interesting form as follows: Let any line be:

$$x \cos \alpha + y \sin \alpha - a = 0$$

If the coördinates of any point on this line be substituted in this equation, the left member reduces to zero. If the coördinates of any point not on the line be substituted for x and y in the equation, the left member of the equation does not reduce to zero, but becomes negative if the given point is on the origin side of the line and positive if the given point is on the non-origin side of the line. The absolute value of the left member in each case gives the distance of the given point from the line. Thus every line may be said to have a "positive side" and a "negative side." The "negative side" is the side toward the origin.

Exercises

1. Find the distance of the point (4, 5) from the line $3x + 4y = 10$.

2. Find the distance from the origin to the line $x/3 - y/4 = 1$.

3. Find the distance from $(-3, -4)$ to:

$$12(x + 6) = 5(y - 2)$$

4. Find the distance from (3, 4) to the line $x/3 - y/4 = 1$.

5. Find the distance between the parallel lines $y = 2x + 3$, $y = 2x + 5$.

6. Find the distance between $y = 2x - 3$, $y = 2x + 5$.

7. Find the distance from (0, 3) to $4x - 3y = 12$.

8. Find the distance from (0, 1) to $x + 2 - 2y = 0$.

250. Tangent to a Circle at a Given Point. The equation of the tangent to the circle obtained in §246 is the equation of the tangent line having a given or required slope m. We shall now find the equation of the line that is tangent to the circle *at a given point* (x_0, y_0).

The line:

$$a = \rho \cos (\theta - \alpha) \tag{1}$$

or its equivalent:

$$x \cos \alpha + y \sin \alpha = a \tag{2}$$

is tangent to the circle of radius a, and the point of tangency is at the end of the radius whose direction angle is α. The point of tangency is therefore $(a \cos \alpha, a \sin \alpha)$. Hence, multiplying (2) through by a, we obtain:

$$x(a \cos \alpha) + y(a \sin \alpha) = a^2 \tag{3}$$

or:

$$\mathbf{x_0 x + y_0 y = a^2} \tag{4}$$

which is the equation of the line tangent at the point (x_0, y_0) to the circle of radius a.

Thus $3x + 4y = 25$ is tangent to $x^2 + y^2 = 25$ at the point $(3, 4)$.

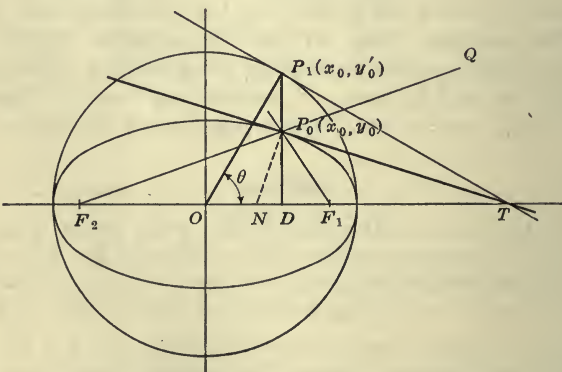


Fig. 178.—Tangent to the Ellipse at a Given Point.

251. Tangent to the Ellipse at a Given Point. It is easy to draw the tangent to the ellipse at any desired point. In Fig. 178, let P_0 be the point at which a tangent is desired. Then draw the major circle, and let P_1 of the circle be a point on the same ordinate as P_0. Draw a tangent to the circle at P_1 and let it meet the x-axis at T. Then when the circle is projected to form the ellipse, the straight line P_1T is projected to make the tangent to the ellipse. Since T when projected remains the same point and since P_0 is the projection of P_1, the line through P_0 and T is the tangent to the ellipse required.

The equation of the tangent P_0T is also readily found. The equation of P_1T is:

$$xx_0 + yy'_0 = a^2 \tag{1}$$

To project this into the line P_0T it is merely necessary to multiply the ordinates y and y'_0 by b/a; that is, to substitute $y = ay/b$ and $y'_0 = ay_0/b$. Whence (1) becomes:

$$x_0x + a^2y_0y/b^2 = a^2 \tag{2}$$

or dividing by a^2,

$$\mathbf{x_0x/a^2 + y_0y/b^2 = 1} \tag{3}$$

which is the tangent to:

$$x^2/a^2 + y^2/b^2 = 1$$

at the point (x_0, y_0).

Exercises

1. Find the equations of the tangents to the ellipse whose semi-axes are 4 and 3 at the points for which $x = 2$.

2. Find the equations of the tangents to $x^2/16 + y^2/9 = 1$ at the ends of the left latus rectum.

3. Required the tangents to $x^2/9 + y^2/4 = 1$ making an angle of 45° with the x-axis.

4. Find the equations of the tangents to $x^2/100 + y^2/25 = 1$ at the points where $y = 3$.

5. Find the equations of the tangents to $x^2/36 + y^2/16 = 1$ at the points where $x = y$.

252. The Tangent, Normal, and Focal Radii of the Ellipse. In the right triangle P_1OT, Fig. 178, the side P_1O is a mean proportional between the entire hypotenuse OT and the adjacent segment OD. That is:

$$a^2 = x_0OT$$

But:

$$\begin{aligned} F_1T &= OT - OF_1 \\ &= a^2/x_0 - ae \end{aligned}$$

Likewise: $F_2T = OT + OF_2$

$$= a^2/x_0 + ae$$

Therefore: $F_1T/F_2T = (a^2/x_0 - ae)/(a^2/x_0 + ae)$

$$= (a - ex_0)/(a + ex_0)$$

But by **§231** this last ratio is equal to r_1/r_2. Therefore we may write: $F_1T/F_2T = P_0F_1/P_0F_2$.

Hence T, which divides the base F_2F_1 of the triangle $P_0F_2F_1$ externally at T in the ratio of the two sides PF_2 and PF_1 of the triangle, lies on the bisector of the external angle F_1P_0Q of the triangle $F_2P_0F_1$. This proves the important theorem:

The tangent to the ellipse bisects the external angle between the focal radii at the point.

This theorem provides a second method of constructing a tangent at a given point of an ellipse, often more convenient than that of **§251**, since the method of **§251** often runs the construction off of the paper.

The normal P_0N, being perpendicular to the tangent, must bisect the internal angle $F_2P_0F_1$ between the focal radii F_2P_0 and F_1P_0.

Since the angle of reflection equals the angle of incidence for light, sound, and other wave motions, a source of light or sound at F_1 is "brought to a focus" again at F_2, because of the fact that the normal to the ellipse bisects the angle between the focal radii.

253. Additional Equations of the Straight Line. The equations of the straight line in the slope form:

$$y = mx + b \tag{1}$$

and in the normal forms:

$$\rho \cos (\theta - \alpha) = a \tag{2}$$

$$x \cos \alpha + y \sin \alpha = a \tag{3}$$

and the general form:

$$ax + by + c = 0 \tag{4}$$

have already been used. Two constants and only two are necessary for each of these equations. The constants in the first equation are m and b; in the second and third, α and a; in the fourth a/c and b/c, or any two of the ratios that result from dividing through by one of the coefficients. Equation (4) appears to contain three constants, but it is only the relative size of these that determines the particular line represented by the equation, since the line would remain the same when the equation is multiplied or divided through by any constant (not zero).

These facts are usually summarized by the statement that two conditions are necessary and sufficient to determine a straight line. The number of ways in which these conditions may be given is, of course, unlimited. Thus a straight line is determined if we say, for example, that the line passes through the vertex of an angle and bisects that angle, or if we say that the line passes through the center of a circle and is parallel to another line, or if we say that the straight line is tangent to two given circles, etc. An important case is that in which the line is determined by the requirement that it pass through a given point in a given direction. The equation of the line adapted to this case is readily found. Let the given point be (x_1, y_1). The line through the origin with the required slope is

$$y = mx$$

Translate this line so that it passes through (x_1, y_1) and we have

$$\mathbf{y - y_1 = m(x - x_1)} \tag{5}$$

Another way of obtaining the same result is: substitute the coördinates (x_1, y_1) in (1):

$$y_1 = mx_1 + b \tag{6}$$

Subtract the members of this from (1) above, so as to eliminate b. There results:

$$y - y_1 = m(x - x_1) \tag{7}$$

This is the required equation; the given point is (x_1, y_1) and the direction of the line through that point is given by the slope m.

Another important case is that in which the straight line is determined by requiring it to pass through two given points. Let the second of the given points be (x_2, y_2). Substitute these coördinates in (5):

$$y_2 - y_1 = m(x_2 - x_1) \tag{8}$$

To eliminate m, divide the members of (7) by the members of (8):

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \tag{9}$$

or, as it is usually written:

$$\frac{\mathbf{y} - \mathbf{y}_1}{\mathbf{x} - \mathbf{x}_1} = \frac{\mathbf{y}_2 - \mathbf{y}_1}{\mathbf{x}_2 - \mathbf{x}_1} \tag{10}$$

This is the equation of a line passing through two given points. Since (10) may be looked upon as a proportion, the equation may be written in a variety of forms.

254. The Circle Through Three Given Points. In general, the equation of a circle can be found when three points are given. Either of the general equations of the circle:

$$(x - h)^2 + (y - k)^2 = a^2 \tag{1}$$

or:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{2}$$

contains three unknown constants, so that in general three conditions may be imposed upon them. It is best to illustrate the general method by a particular example. Let the three given points be $(-1, 3)$, $(0, 2)$, and $(5, 0)$. Then since the coördinates of these points must satisfy the equation of the circle, we obtain from (2) above:

$$1 + 9 - 2g + 6f + c = 0 \tag{3}$$

$$4 + 4f + c = 0 \tag{4}$$

$$25 + 10g + c = 0 \tag{5}$$

Eliminating c from (3) and (4) and from (4) and (5), we obtain:

$$6 - 2g + 2f = 0$$

$$21 + 10g - 4f = 0$$

Eliminating f:

$$g = -5\tfrac{1}{2}$$

whence:

$$f = -8\tfrac{1}{2}$$

and

$$c = 30$$

So the equation of the circle is:

$$x^2 + y^2 - 11x - 17y + 30 = 0$$

Exercises

1. Find the equation of the line passing through $(2, 3)$ with slope $2/3$.

2. Find the equation of the line passing through $(2, 3)$, $(3, 5)$.

3. Find the line passing through $(2, -1)$ making an angle whose tangent is 2 with the x-axis.

4. Find the line through $(2, 3)$ parallel to $y = 7x + 11$.

5. A line passes through $(-1, -3)$ and is perpendicular to $y - 2x = 3$. Find its equation.

6. Find the line passing through $(-2, 3)$, $(-3, -1)$.

7. Find the equation of the line which passes through $(-1, -3)$, $(-2, 4)$.

8. Find the slope of the line that passes through $(-1, 6)$, $(-2, 8)$.

9. Find the equation of the line passing through the left focus and the upper end of the right latus rectum of $x^2/25 + y^2/9 = 1$.

10. Find the equation of the circle passing through $(2, 8)$, $(5, 7)$, and $(6, 6)$.

11. Find the equation of the circle which passes through $(1, 2)$, $(-2, 3)$, and $(-1, -1)$.

12. Find the equation of the parabola in the form $y^2 = 4px$ which passes through the point $(2, 4)$.

255. Change from Polar to Rectangular Coördinates. The relations between x, y of the Cartesian system and ρ, θ of the polar system have already been explained and use made of them. The relations are here brought together for reference:

$$\mathbf{x} = \rho \cos \theta \qquad (1)$$

$$\mathbf{y} = \rho \sin \theta \qquad (2)$$

By these we may pass from the Cartesian equation of any locus to the equivalent polar equation of that locus. Dividing (2) by (1) and also squaring and adding, we obtain:

$$\theta = \tan^{-1} \mathbf{y}/\mathbf{x} \qquad (3)$$

$$\rho = \sqrt{\mathbf{x}^2 + \mathbf{y}^2} \qquad (4)$$

These may be used to convert any polar equation into the Cartesian equivalent.

256. Rotation of Any Locus. It has already been explained that any locus can be rotated through an angle α by substituting $(\theta_1 - \alpha)$ for θ in the polar equation of the locus. It remains to determine the substitutions for x and y which will bring about the rotation of a locus in rectangular coördinates. Let us consider any point P of a locus before and after rotation through the given angle α. Call the coördinates of the point before rotation

(x, y) in rectangular coördinates and (ρ, θ) in polar coördinates. Then, from (1) and (2), **§255,**

$$x = \rho \cos \theta \quad (1)$$
$$y = \rho \sin \theta \quad (2)$$

Call the coördinates of the point after rotation (x_1, y_1) and (ρ_1, θ_1), but note that the value of ρ is unchanged by the rotation. Then for the point P', Fig. 179, we may write:

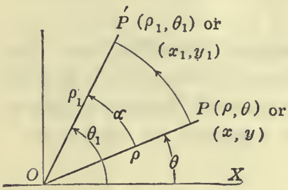


Fig. 179.—Rotation of Any Locus.

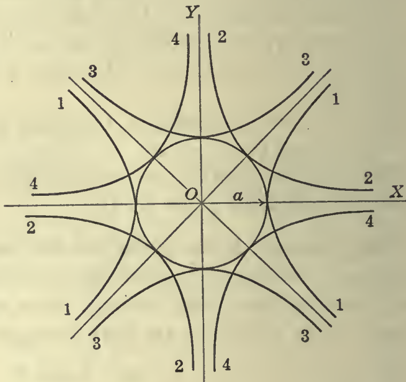


Fig. 180.—Effect of Rotation on the Special Forms $x^2 + y^2$, $2xy$, and $x^2 - y^2$.

$$x_1 = \rho \cos \theta_1 \quad (3)$$
$$y_1 = \rho \sin \theta_1 \quad (4)$$

Since, however, the rotation requires that

$$\theta = \theta_1 - \alpha \quad (5)$$

equations (1) and (2) become:

$$x = \rho \cos (\theta_1 - \alpha) = \rho \cos \theta_1 \cos \alpha + \rho \sin \theta_1 \sin \alpha \quad (6)$$
$$y = \rho \sin (\theta_1 - \alpha) = \rho \sin \theta_1 \cos \alpha - \rho \cos \theta_1 \sin \alpha \quad (7)$$

But, from (3) and (4), $\rho \cos \theta_1$ and $\rho \sin \theta_1$ are the new values of x and y; hence, substituting in (6) and 7) from (3) and (4) we obtain:

$$\mathbf{x} = \mathbf{x}_1 \cos \alpha + \mathbf{y}_1 \sin \alpha \quad (8)$$
$$\mathbf{y} = \mathbf{y}_1 \cos \alpha - \mathbf{x}_1 \sin \alpha \quad (9)$$

Hence if the equation of any locus is given in rectangular co-ordinates, it is rotated through the positive angle α by the substitutions

$$\begin{aligned} &\mathbf{x} \cos \alpha + \mathbf{y} \sin \alpha \text{ for } \mathbf{x} \\ &\mathbf{y} \cos \alpha - \mathbf{x} \sin \alpha \text{ for } \mathbf{y} \end{aligned} \tag{10}$$

in which it is permissible to drop the subscripts, if the context shows in each case whether we are dealing with the old x and y or with the new x and y.

If the required rotation is clockwise, or negative, we must replace α by $(-\alpha)$ in all of the above equations.

Whenever convenient, the equation of a curve should be taken in the polar form if it is required to rotate the locus.

Important Facts: The following facts should be remembered by the student:

(1) *To rotate a curve through* 90°, *change x to y and y to $(-x)$.* This fact has been noted in §68.

(2) *Rotation through any angle leaves the expression $x^2 + y^2$ (or any function of it) unchanged.* This is obvious since the circle $x^2 + y^2 = a^2$ is not changed by rotation about (0, 0).

(3) *Rotation through* $+45°$ *changes* $2xy$ *to* $y^2 - x^2$.
Rotation through $-45°$ *changes* $2xy$ *to* $x^2 - y^2$.

(4) *Rotation through* $+45°$ *changes* $x^2 - y^2$ *to* $2xy$.
Rotation through $-45°$ *changes* $x^2 - y^2$ *to* $-2xy$.

Statements (3) and (4) follow at once from consideration of the equations

$$x^2 - y^2 = a^2 \tag{1}$$
$$2xy = a^2 \tag{2}$$
$$y^2 - x^2 = a^2 \tag{3}$$
$$-2xy = a^2 \tag{4}$$

of the four hyperbolas bearing corresponding numbers (1), (2), (3), (4) in Fig. 180. The proper change in any case can be remembered by thinking of the four hyperbolas of this figure.

(5) *The degree of an equation of a locus cannot be changed by a rotation.* This follows at once from the fact that the equations of transformation (8) and (9) are linear.

Exercises

In order to shorten the work, use statements (1) to (4) whenever possible.

1. Turn the locus $x^2 - y^2 = 4$ through 45°.

2. Turn $x^2 + y^2 = a^2$ through 79°. Turn $4xy = 1$ through 45°.

3. Turn $x \cos \alpha + y \sin \alpha = a$ through an angle β. (Since this locus is well known in the polar form, transformation formulas (6) and (7) above may be avoided.)

4. Rotate $x^2 - y^2 = 1$ through 90°.

5. Rotate $x^2 - y^2 = a^2$ through $-45°$.

6. Change the equation $(x - a)^2 + (y - b)^2 = r^2$ to the polar form.

7. Change $\rho \cos 2\theta = 2a$, one of a class of curves known as Cote's spirals, to the Cartesian form.

8. Write the equation of the lemniscate in the polar form.

9. Show that $\rho^2 - 2\rho\rho_1 \cos (\theta - \theta_1) + \rho_1{}^2 = a^2$ is the polar equation of a circle with center at (ρ_1, θ_1) and of radius a.

10. Write the Cartesian equation of the locus $\rho^2 = 16 \sin 2\theta$.

11. Turn $\rho^2 = 8 \sin 2\theta$ through an angle of 45°.

12. Rotate $x^2 - 2y^2 = 1$ through 90°.

13. Rotate $(x^2 + y^2)^{3/2} + (x^2 - y^2)^{3/2} = 1$ through 45°.

14. Rotate $\log (x^2 + y^2) = \tan (x^2 - y^2)$ through 45°.

257. Ellipse with Major Axis at 45° to the OX Axis. The ellipse frequently arises in applied science as the resultant of the projection of the motion of two points moving uniformly on two circles, as has already been explained in **§186.** Thus the parametric equations:

$$x = a \cos t \tag{1}$$

$$y = b \sin t \tag{2}$$

define an ellipse which may be considered the resultant of two S.H.M. in quadrature. We shall prove that the equations:

$$x = a \cos t \tag{3}$$

$$y = a \sin (t + \alpha) \tag{4}$$

define an ellipse, with major axis making an angle of 45° with OX.

The graph is readily constructed as in Fig. 181. The Cartesian equation of the curve is found by eliminating t between (3) and (4). Expanding the sin $(t + \alpha)$ in (4) and substituting from (3) we obtain:

$$y = x \sin \alpha + \sqrt{a^2 - x^2} \cos \alpha \tag{5}$$

Transposing and squaring:

$$x^2 - 2xy \sin \alpha + y^2 = a^2 \cos^2 \alpha \tag{6}$$

By **§256** rotate the curve through an angle of $(-45°.)$ We know that $(x^2 + y^2)$ is unchanged and that $2xy$ is to be replaced by $(x^2 - y^2)$. Therefore (6) becomes:

$$x^2(1 - \sin\alpha) + y^2(1 + \sin\alpha) = a^2\cos^2\alpha \qquad (7)$$

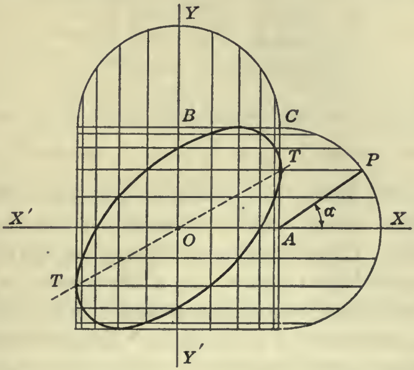


FIG. 181.—The Ellipse $x = a\cos t,\ y = a\sin(t + \alpha)$.

Replacing $\cos^2\alpha$ by $1 - \sin^2\alpha$, and dividing through by the right member, we obtain:

$$\frac{x^2}{a^2(1 + \sin\alpha)} + \frac{y^2}{a^2(1 - \sin\alpha)} = 1 \qquad (8)$$

which may be written:

$$\frac{x^2}{2a^2\cos^2\frac{\beta}{2}} + \frac{y^2}{2a^2\sin^2\frac{\beta}{2}} = 1 \qquad (9)$$

where β is the complement of α. Equation (8) or (9) proves that the locus is an ellipse. It is *any* ellipse, since by properly choosing a and α the denominators in (8) can be given any desired values. Hence the pair of parametric equations (3) and (4), or the Cartesian equation (5) represents an ellipse with its major axis inclined $+ 45°$ to the OX-axis.

258. General Equation of the Second Degree. The general equation of the second degree in two variables may be written in the standard form:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (1)$$

In the next two sections we shall show that the *general equation of the second degree in two variables represents a conic.* We shall be able to distinguish three cases as follows:

The general equation of the second degree represents:

$$\text{an ellipse if } h^2 - ab < 0 \qquad (2)$$
$$\text{a parabola if } h^2 - ab = 0 \qquad (3)$$
$$\text{a hyperbola if } h^2 - ab > 0 \qquad (4)$$

To render the above classification true in all cases we must classify the "imaginary ellipse," $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$, as an ellipse, and other degenerate cases must be similarly treated. The expression $h^2 - ab$ is called the **quadratic invariant** of the equation (1), so called because its value remains unchanged as the curve is moved about in the coördinate plane. In other words, as the locus (1) is translated or rotated to any new position in the plane, and while of course the coefficients of x^2, xy, and y^2 change to new values, the function of these coefficients, $h^2 - ab$, does *not* change value, but remains invariant. This fact is not proved in this book, but it can readily be proved by comparing the value of $h^2 - ab$ before and after the substitutions:

$$x \cos \alpha + y \sin \alpha - m \text{ for } x$$
$$y \cos \alpha - x \sin \alpha - n \text{ for } y$$

where m and n indicate the amount of the translation, and α the angle of rotation.

259.* Conics with Their Axes Parallel to the Coördinate Axes.[1] Let us consider the equation

$$ax^2 + by^2 + 2gx + 2fy + c = 0 \qquad (1)$$

If we solve this equation for y in terms of x, we get

$$y = \frac{-f \pm \sqrt{-abx^2 - 2bgx - bc + f^2}}{b} \qquad (2)$$

1. We saw in completing the squares, §**77**, that (1) is the equation of an ellipse when a and b are alike in algebraic signs. We can now restate this condition by saying that (2) is the equation of an ellipse when the coefficient of x^2 is negative. Note

[1] §§**259** and **260** are from the correspondence course prepared by Professor H. T. Burgess.

that the equation of a circle is included as a special case when $a = b$.

2. We saw in completing the squares, **§87,** that (1) is the equation of a hyperbola when a and b have unlike signs. We can restate this condition by saying that (2) is the equation of a hyperbola when the coefficient of x^2 is positive.

3. We observe that (1) is the equation of a parabola when $a = 0$. We can restate this condition by saying that (2) is the equation of a parabola when the coefficient of x^2 is zero.

260.* The General Case. Write the quadratic in two variables in the standard form:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

I. We have already seen, **§43,** that when and only when $h = 0$ and $a = b$ the locus of (1) is a circle.

II. When h is not equal to zero, we have as yet no knowledge of the nature of the locus represented by (1), except that it is not a circle.

Let us rotate this locus clockwise through an angle α and see if the equation can be simplified so that the character of the locus represented by (1) can be recognized. Substituting in (1) from **§256,** we get

$$\begin{aligned} a(x \cos \alpha - y \sin \alpha)^2 &+ 2h(x \cos \alpha - y \sin \alpha)(x \sin \alpha + y \cos \alpha) \\ &+ b(x \sin \alpha + y \cos \alpha)^2 + 2g(x \cos - y \sin \alpha) \\ &+ 2f(x \sin \alpha + y \cos \alpha) + c = 0 \end{aligned} \tag{2}$$

If we simplify (2), we find that the coefficient of the term in xy is:

$$2(b - a) \sin \alpha \cos \alpha + 2h(\cos^2 \alpha - \sin^2 \alpha) \tag{3}$$

This term will drop out of (2), if we can find a value for the angle α that will make (3) zero.

Substituting in (3) from equations (1) and (2), **§165,** we get:

$$(b - a) \sin 2\alpha + 2h \cos 2\alpha = 0 \tag{4}$$

From this we find:

$$\tan 2\alpha = \frac{2h}{a - b} \tag{5}$$

Hence if we choose α as half of the angle whose tangent is $\frac{2h}{a-b}$, equation (2) will have no term in xy, and it will be of the form:

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0 \tag{6}$$

where A, B, G, F, and C stand for long expressions in terms of the coefficients of equation (1).

Since the loci of equations (1) and (6) are identically the same curve, we now see from §258 that the locus of (1) must be an ellipse, hyperbola, or parabola.

III. We can now devise a test by which we can tell immediately which curve is represented by equation (1). If we solve (1) for y in terms of x, we get

$$y = \frac{-(hx+f) \pm \sqrt{(h^2-ab)x^2 + 2(hf-gb)x + f^2 - bc}}{b} \tag{7}$$

Let us now consider the two equations:

$$y_1 = -\frac{h}{b}x - \frac{f}{b} \tag{8}$$

$$y_2 = \pm\frac{\sqrt{(h^2-ab)x^2 + 2(hf-gb)x + f^2 - bc}}{b} \tag{9}$$

It is obvious that the locus of (7), whatever it is, may be obtained by shearing the locus of (9) in the line (8). We must consider the three following cases:

1. When $h^2 < ab$ the coefficient of x^2 in (9) is negative and the locus of (9) is an ellipse. Hence the locus of (7) is a locus made by shearing an ellipse in a line, and is therefore a closed curve. The locus of (7) is in this case an ellipse, for it must be either an ellipse, a hyperbola, or a parabola by II, and it cannot be either a hyperbola or a parabola since it is a closed curve.

2. When $h^2 > ab$ the coefficient of x^2 in (9) is positive and the locus of (9) is a hyperbola. Hence the locus of (7) is a locus made by shearing a hyperbola in a line, and is therefore an open curve with two branches. The locus of (7) is in this case a hyperbola, for it cannot be an ellipse or a parabola since it has two open branches.

3. When $h^2 = ab$ the coefficient of x^2 in (9) is zero and the locus of (9) is a parabola. Hence the locus of (7) is a locus made by shearing a parabola in a line, and is therefore an

open curve with one branch. The locus of (7) is in this case a parabola, for it cannot be an ellipse or a hyperbola since it has one open branch.

We now state the results in this form: The locus of the general equation of the second degree in two variables is for

$$h^2 < ab \quad \text{an ellipse}$$
$$h^2 > ab \quad \text{a hyperbola}$$
$$h^2 = ab \quad \text{a parabola}$$

If we shear the locus of (7) in any line $y = mx + b$, the form of the equation is not changed. Hence the following important facts:

The shear of an ellipse in a line is an ellipse.

The shear of a hyperbola in a line is a hyperbola.

The shear of a parabola in a line is a parabola.

If we put (mx) for x and (ny) for y in (7), no change will be made in the sign of the coefficient of x^2; *hence the elongation or contraction (orthographic projection) of an ellipse, hyperbola, or parabola in any direction is an ellipse, hyperbola, or parabola.*

261. Shear of the Circle. The effect of the addition of the term mx to $f(x)$, in the equation $y = f(x)$, has been shown in **§37** to

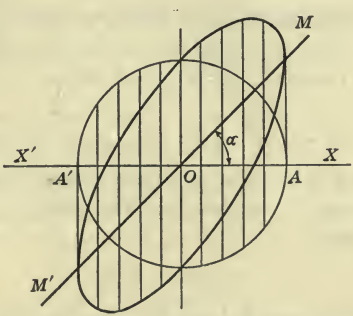


FIG. 182.—The Ellipse Looked Upon as the Shear of a Circle OA in a Line $M'OM$.

be to change the shape of the locus by lamellar or shearing motion of the xy plane. We usually speak of this process as "the shear of the locus $y = f(x)$ in the line $y = mx$." When applied to the circle

$y = \pm \sqrt{a^2 - x^2}$ the effect is to move vertically the middle point of each double ordinate of the circle to a position on the line $y = mx$. The result of the shearing motion is shown in Fig. 182. *The area bounded by the curve is unchanged by the shear.*

The equation after shear is:

$$y = mx \pm \sqrt{a^2 - x^2} \tag{1}$$

This is the same form as equation (5) of **§257,** if we put $m = \dfrac{\sin \alpha}{\cos \alpha}$ and then multiply all ordinates by $\cos \alpha$. Therefore the curve of Fig. 182 is an ellipse.

The straight line $y = mx$ passes through the middle points of the parallel vertical chords of the ellipse

$$y = mx + \sqrt{a^2 - x^2} \tag{2}$$

The locus of the middle points of parallel chords of any curve is called a **diameter** of that curve. We have thus shown that the diameter of the ellipse is a straight line. Since the same reasoning applies to

$$y = mx + (b/a)\sqrt{a^2 - x^2} \tag{3}$$

which may be regarded as any ellipse in any way oriented with respect to the origin, the proof shows that the mid-points of arbitrarily selected parallel chords of an ellipse is always a straight line.

262. A Second Proof. The generality of the preceding fact may seem clearer if the ellipse be kept fixed in position while the direction of the set of parallel chords is arbitrarily selected. Consider first the circle

$$x^2 + y^2 = a^2 \tag{1}$$

and draw any set of parallel chords. Let the slope of these chords be s. Then the equation of the chords is

$$y = sx + p \tag{2}$$

in which p is an arbitrary parameter, to various values of which correspond the different chords of the family of parallel chords.

The equation of the bisectors of all of the chords is a line through the origin perpendicular to (2), or:

$$y = -\frac{x}{s} \tag{3}$$

Now if the circle (1) and the chords (2) and the diameter (3) be changed by orthographic projection upon a plane through the X-axis, then the circle (1) becomes an ellipse, while the parallel chords and the line through their mid-points remain straight lines, but with modified slopes. Let the given orthographic projection multiply all ordinates of (1), (2) and (3) by $\frac{b}{a}\cdot$ Then the equation of the ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4}$$

The parallel chords now have the equation

$$y = \frac{sbx}{a} + \frac{bp}{a} \tag{5}$$

The equation of the locus of the mid-points of the parallel chords or the diameter is:

$$y = -\frac{bx}{sa} \tag{6}$$

Representing $\frac{sb}{a}$, the slope of (5), by m, equation (6) takes the form:

$$y = -\frac{b^2x}{ma^2} \tag{7}$$

which is the equation of the diameter of (4) that bisects the family of parallel chords of slope m.

263. Confocal Conics. Fig. 183 shows a number of ellipses and hyperbolas possessing the same foci A and B. This family of curves may be represented by the single equation:

$$\frac{x^2}{a^2 - k} + \frac{y^2}{b^2 - k} = 1 \tag{1}$$

in which the parameter k takes on any value lying between 0 and a^2, and in which $a > b$. If k satisfies the inequality:

$$0 < k < b^2$$

the curves are ellipses. If k satisfies the inequality:

$$b^2 < k < a^2$$

the curves are hyperbolas. The ellipses of Fig. 183 may be regarded as representing the successive positions of the wave front of sound waves leaving the sounding body AB; or they may be regarded as the equipotential lines around the magnet AB, of which the hyperbolas represent the lines of magnetic force.

Exercises

1. Sketch the curve:

$$y = 2x + \sqrt{4 - x^2}.$$

2. Draw the curve:

$$x = 2 \cos \theta$$
$$y = 2 \sin (\theta + \pi/6).$$

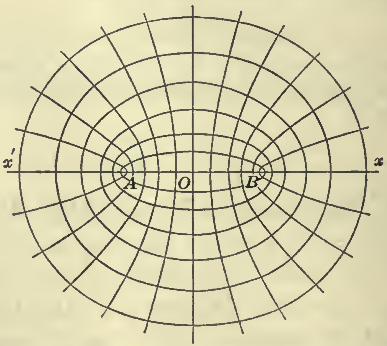


Fig. 183.—Confocal Ellipses and Hyperbolas. Note that the curves of one class cut those of the other class *orthogonally*.

3. Find the axes of the ellipse:

$$x = 3 \cos \theta$$
$$y = 3 \sin (\theta + \pi/4).$$

4. Draw the curve:

$$y = x \pm \sqrt{6x - x^2}.$$

5. Draw the curve:

$$y = x \pm \sqrt{x^2 - 6x}.$$

6. Show that:

$$y = x \pm \sqrt{6x} \text{ is a parabola.}$$

7. Sketch the curve:

$$y = (1/2)x + \sqrt{16 - x^2}.$$

8. Sketch the curve:

$$y = 5x \sin 60° + \cos 60° \sqrt{25 - x^2}.$$

9. Discuss the curve:

$$x^2/a^2 + y^2/b^2 - 2(xy/ab) \cos \alpha = \sin^2 \alpha.$$

Show that the locus is always tangent to the rectangle $x = \pm a$, $y = \pm b$, and that the points of contact from a parallelogram of constant perimeter $4\sqrt{a^2 + b^2}$ for all values of α.

10. Show that $x = a \cos (\theta - \alpha)$, $y = b \cos (\theta - \beta)$ represents an ellipse for all values of a and β.

11. Prove from equation (13), **§257**, that the distance from the end of the minor to the end of the major axis of the resulting ellipse remains the same independently of the magnitude of α.

12. Show that the following construction of the hyperbola $xy^2 = a^3$ is correct. On the $-$ x-axis lay off $OC = a$. Connect C with any point A on the y-axis. At C construct a perpendicular to AC cutting the y-axis in B. At B erect a perpendicular to BC cutting the $+$ x-axis at D. Through A draw a parallel to the x-axis and through D draw a parallel to the y-axis. The two lines last drawn meet at P, a point on the desired curve.

13. Explain the following construction of the cubical parabola $a^2y = x^3$. Lay off OB on the $-$ y-axis equal to a. From B draw a line to any point C of the x-axis. At C erect a perpendicular to BC cutting the y-axis at D. At D erect a perpendicular to CD cutting the x-axis at E. Lay off OE on the y-axis. Then OE is the ordinate of a point of the curve for which the abscissa is OC.

14. Explain and prove the following construction of the semi-cubical parabola, $ay^2 = x^3$. Lay off on the $-$ x-axis $OA = a$. From A draw a parallel to the line $y = mx$, cutting the y-axis in B. Erect at B a perpendicular to AB cutting the x-axis at C, and at C erect a perpendicular to OC. The point of intersection with $y = mx$ is a point of the curve.

Problems for Review

1. Find the approximate equations for the following data:
(*a*) Steam pressure: v = volume, p = pressure.
(*b*) Gas-engine mixture: v = volume, p = pressure.

(*a*) v	(*a*) p	(*b*) v	(*b*) p
2	68.7	3.54	141.3
4	31.3	4.13	115.0
6	19.8	4.73	95.0
8	14.3	5.35	81.4
10	11.3	5.94	71.2
		6.55	63.5
		7.14	54.6

2. Show that $\rho^2 = a^2 \cos 2\theta$ is the polar equation of a lemniscate.

3. When an electric current is cut off, the rate of decrease in the current is proportional to the current. If the current is 36.7 amperes when cut off and decreases to 1 ampere in one-tenth of a second, determine the relation between the current C and the time t.

4. Write four other equations for the circle $\rho = 2\sqrt{3} \sin \theta - 2 \cos \theta$.

5. Write four other equations for the sinusoid $y = \sin x - \sqrt{3} \cos x$.

6. Find the angle that $3x + 4y = 12$ makes with $4x - 3y = 12$.

7. From the equation

$$\theta = 6 \sin (2t - 1°)$$

determine the amplitude, period, and frequency of the S.H.M.

8. A simple sinusoidal wave has a height of 3 feet, a length of 29 feet, and a velocity of 7 feet per minute. Another wave with the same height, length, and velocity lags 15 feet behind it. Give the equation of each.

9. Interpret $r_1(\cos \theta_1 + i \sin \theta_1)$ as an operator upon $r_2(\cos \theta_2 + i \sin \theta_2)$.

10. Give a rule for writing down the value of i^n.

11. Calculate:

$$\frac{(3\sqrt{3} - 3i)^2(-1 + \sqrt{3}i)^3}{(2 + 2\sqrt{2}i)} + \frac{(\cos 36° + i \sin 36°)(\cos 20° + i \sin 20°)}{2(\cos 11° + i \sin 11°)}.$$

12. Calculate: $(1 - \sqrt{3}i)^{2/3}$.

13. Write the inverse functions of the following:

(*a*) $y = ax^n$, (*b*) $y = \sin x$, (*c*) $y = e^x$, (*d*) $y = \log_e x$.

14. Plot the amount of tin required to make a tomato can to hold 1 quart as a function of the radius of its base. Determine approximately from the graph the dimensions requiring the least tin.

15. Find the axes of the ellipse whose foci are (2, 0) and (− 2, 0), and whose directrices are $x = \pm 5$.

16. Write the polar equation for the ellipse in problem 15.

17. Find the equation of the hyperbola whose foci are (5, 0) and (− 5, 0), and whose directrices are $x = \pm 2$.

18. Write the equation of the hyperbola in 17 in polar coördinates.

19. Discuss the curve $\rho(1 + \cos\theta) = 4$. Write its equation in rectangular coördinates.

20. Find the foci of the hyperbola $2xy = a^2$. Also its eccentricity.

21. What property of the parabola is useful in designing automobile headlights?

22. How do you draw a tangent to an ellipse? To a parabola?

23. Find the equation of a point whose distance from the point (3, 4) is always twice its distance from the line $3x + 4y = 12$. What is the locus?

24. Give the type of each of the following conics:

(*a*) $2x^2 + 2y^2 + 3x - 4y + 3 = 0.$
(*b*) $x^2 + 4xy + 4y^2 + x - 3y + 8 = 0.$
(*c*) $x^2 + 3xy - 3y^2 + 3x - 2y - 3 = 0.$
(*d*) $x^2 - 5xy + 7y^2 + 2x + 3y + 28 = 0.$

25. Solve each of the equations in problem 24 for y and explain how the graphs may be constructed by shear.

26. A point moves so that the quotient of its distance from two fixed points is a constant. Find the equation of the locus of the point.

27. Evaluate:

$$\log 10 - \log_2 8 + \log_7 49^2.$$

28. Find the maximum and minimum value of $(3 \sin x - 4 \cos x)$. What values of x give these maximum and minimum values?

29. Find the equation of a circle passing through the points (1, 2), (−1, 3) and (3, − 2).

30. A sinusoidal wave has a wave-length of π, a period of π, and an amplitude of π. Write its equation.

31. Compute graphically the following:

$$(1 + i)(1 - i); \quad (1 + i) + (1 - i);$$

$$\frac{3+7i}{7+3i}; \quad 7 \text{ cis } 47° \times 6 \text{ cis } (-14°);$$

$$(7+6i)^{27}; \quad \sqrt[15]{7i+31}.$$

32. Prove by the addition formulas that:

$$\begin{aligned} \sin(90° - \tau) &= \cos\tau \\ \sin(90° + \tau) &= \cos\tau \\ \sin(360° - \tau) &= -\sin\tau\,. \\ \tan(\tau + 270°) &= -\cot\tau. \end{aligned}$$

33. Sketch the curves:

$$\begin{aligned} y &= 2^x \\ y &= 3^x \\ y &= 2^{0.63x}. \end{aligned}$$

What property of the exponential function do these curves illustrate?

34. Sketch $y = 2^x$ and $y = 3^x$.

35. Solve: $x^2 + 6x + \sqrt{x^2 + 6x + 1} = 1$.

36. Find graphically the product of $3 - 2i$ by $-2 + i$.

37. Find all the values of:

$$(\cos\theta + i\sin\theta)^3; \ (\cos\theta + i\sin\theta)^{1/4}; \ \sqrt[8]{1}; \ \sqrt[5]{1}.$$

38. Write a short theme on operators, making mention of (*a*) the integers; (*b*) (-1); (*c*) $\sqrt{-1}$; (*d*) cis θ. Develop the rules for addition, subtraction, multiplication, and division of vectors, and state them in systematic form.

39. Show that

$$\sin(a + b + c) = \sin a \cos b \cos c + \cos a \sin b \cos c$$
$$+ \cos a \cos b \sin c - \sin a \sin b \sin c.$$

40. Sketch the curves

$$\begin{aligned} y &= 3^x; \\ \frac{y}{2} &= 3^x; \\ y &= 3^{x + 0.63}. \end{aligned}$$

on the same sheet of paper. What property of the exponential function do these curves illustrate?

41. Draw upon squared paper, using 2 cm. = 1, the curve $y^2 = x$. By counting the small squares of the paper find the area bounded by the curve and the ordinates $x = 1/2, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, \ldots$ By plotting these points upon some form of coördinate paper, find the functional relation existing between the x coördinate and the area under the curve.

42. The latitude of two towns is 27° 31′. They are 7 miles apart measured on the parallel of latitude. Find their difference in longitude.

43. Solve $3^{x^2-1} = 2^{x+1}$. Be very careful to take account of all questionable operations . There are two solutions.

44. Find (three problems) the equation connecting:

x	y	x	y	x	y
6.8	19.0	002	18.8		
14.2	21.6	0.04	19.3		
21.8	23.2	0.06	19.6		
32.0	26.3	0.08	19.7		
46.5	31.5	0.10	20.2	1.3	21
65.0	39.1	0.12	20.5	2.0	25
78.0	47.0	0.14	21.0	2.8	29
		0.16	21.4	3.7	33
		0.18	21.8	4.3	35
		0.20	21.9	5.3	38
		0.22	22.1		

45. Find the wave length, period, frequency, amplitude and velocity for

$$y = 10 \sin (2x - 3t).$$

46. Prove that:

$$\frac{\csc^2 A}{\csc^2 A - 2} = \sec 2A.$$

47. Find the parametric equations of the cycloid.

48. Find the equation of the ellipse, center at the origin, axes coinciding with coördinate axes, passing through the point $(-3, 5)$ and having eccentricity 3/5.

49. Define the "logarithm of a number."

50. Prove:

$$\csc 2x(1 - \cos 2x) = \sin x \sec x.$$
$$\csc x(1 - \cos x) = ?$$

51. A S.H.M. has amplitude 6, period 3. Write its equation if time be measured from the *negative* end of the oscillation. State the difference between a S.H.M. and a wave.

52. Find by inspection one value of x satisfying the following equations:

(a) $\cos 45° \cos (90° - x) - \sin 45° \sin (90° - x) = \cos x.$

(b) $\cos (45° - x) \cos (45° + x) - \sin (45° - x) \sin (45° + x) = \cos x.$

53. Sketch on Cartesian paper:

$$\begin{array}{ll} y = 1^x & \\ y = 2^x & y = \log_2 x \\ y = 3^x & y = \log_3 x \\ y = 5^x & y = \log_5 x \\ y = 10^x & y = \log_{10} x. \end{array}$$

54. Solve $3^x + 2x = 1$.

55. Sketch:

$$\begin{array}{llll} \rho = a, & \rho = \sec \theta, & \rho = a \sin \theta, & \rho = - a \sin \theta. \\ \rho = \frac{1}{a}, & \rho = a \cos \theta, & \rho = - a \cos \theta, & \rho = a - a \cos \theta. \\ \rho = 2(- \cos \theta), & & \rho = 2 \cos \theta - 3, & \rho = \cos \theta + \sin \theta. \end{array}$$

56. Simplify the expression:

$$\sin \left(\frac{\pi}{4} - \tau\right) \sec \left(\frac{\pi}{4} + \tau\right) - \sin \left(\frac{\pi}{4} + \tau\right) \sec \left(\frac{\pi}{4} - \tau\right)$$

57. A point moves so that the product of its distance from two fixed points is a constant. Find the equation of the locus. Discuss the curve.

58. Simplify and represent graphically:

$$(a)\ \frac{1 + i}{1 - i},\quad (b)\ (1 + i)(1 + 2i).$$

59. Find the velocity and frequency of the wave of problem 30.

60. Find the coördinates of the center, the eccentricity, and the lengths of the semi-axes of: (a) $x^2 + 3x + y^2 = 7$, (b) $x^2 + 2x + 4y^2 - 3y = 0$, (c) $x^2 - x - y^2 - y = 0$, (d) $x^2 + x + y + 3 = 0$.

61. Find the amplitude, period, frequency and epoch of the following S.H.M.:

$$y = 7 \sin 6t.$$
$$y = 6 \sin 2\pi t.$$
$$y = a \sin (\omega t + \sigma).$$

62. Find $\text{cis}^5\, \theta$. Hence show that:

$$\cos 5x = \cos^5 x - 10 \cos^3 x \sin^2 x + 5 \cos x \sin^4 x.$$

63. Find graphically (on form $M3$) the fifth roots of 2^5 cis 35°.

64. Complete the following equations:

$$\begin{array}{ll} \sin (a \pm b) = ? & \tan 2x = ? \\ \cos (a \pm b) = ? & \cot 2x = ? \end{array}$$

$$\tan(a \pm B) = ? \qquad \sin \frac{\tau}{2} = ?$$

$$\sin 2x = ? \qquad \cos \frac{\tau}{2} = ?$$

$$\cos 2x = ? \qquad \cot \frac{\tau}{2} = ?$$

65. Change the equations of exercises 33 and 40 to logarithmic form. What properties of logarithms are illustrated by these equations?

66. Solve $x^4 - x^3 + 7x + 6 = 0$.

67. Show that the sum of the two focal radii of the ellipse is constant.

68. $y = -3t^2 + 4t - 5$ and $x = 5t$ are the parametric equations of a curve. Discuss the curve.

69. Show that $[r(\cos\theta + i\sin\theta)]\,[r'(\cos\theta' + i\sin\theta')] = rr'[\cos(\theta + \theta') + i\sin(\theta + \theta')]$.

70. Two S.H.M. have amplitude 6 and period two seconds. The point executing the first motion is one-fourth of a second in advance of the point executing the second motion. Write the equations of motion.

71. Show that:

$$\sin 5x = \sin^5 x - 10\sin^3 x \cos^2 x + 5\sin x \cos^4 x.$$

72. Prove that:

$$\tan(45° + \tau) - \tan(45° - \tau) = \frac{4\tan\tau}{1 - \tan^2\tau}.$$

73. Show that the difference of the two focal radii for the hyperbola is constant.

74. Find graphically the quotient of $6 - 2i$ by $3 + 75i$.

75. Solve by inspection, for y:

$$\sin(90° + \tfrac{1}{2}y)\cos(90° - \tfrac{1}{2}y) + \cos(90° + \tfrac{1}{2}y)\sin(90° - \tfrac{1}{2}y) = \sin y.$$

76. Write the parametric equations for the circle, the ellipse, the hyperbola.

CHAPTER XIV

A REVIEW OF SECONDARY SCHOOL ALGEBRA

300. Only the most important topics are included in this review. From five to ten recitations should be given to this work before beginning regular work in Chapter I.

With the kind permission of Professor Hart, a number of the exercises have been taken from the *Second Course in Algebra*, by Wells and Hart.

301. Special Products. The following products are fundamental:

(1) *The product of the sum and difference of any two numbers*:

$$(x + y)(x - y) = x^2 - y^2$$

(2) *The square of a binomial:*

$$(x \pm y)^2 = x^2 \pm 2xy + y^2$$

If the second term of the binomial has the sign $(-)$, then the middle term of the square has the sign $(-)$.

(3) *The product of two binomials having a common term:*

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

thus

$$\begin{aligned}(x + 5)(x - 11) &= x^2 + (5 - 11)x + 5(-11)\\ &= x^2 - 6x - 55\end{aligned}$$

(4) *The product of two general binomials:*

$$(ax + b)(cx + d) = acx^2 + (bc + ad)x + bd$$

thus

$$\begin{aligned}(3a - 4b)(2a + 7b) &= (3a)(2a) + (-8 + 21)ab + (-4b)(7b)\\ &= 6a^2 + 13ab - 28b^2\end{aligned}$$

Exercises

Find mentally the following products:

1. $(5x - 2y)^2$.

2. $(a + 11b)(a + 3b)$.

3. $(a - 2v)(a + 12v)$.

4. $(2m + 3)(m + 4)$.

5. $(y^2 + 4z)(y^2 + 4z)$.

6. $(3xy - 7)^2$.

7. $(3u^2v - 4)(3u^2v + 4)$.
8. $(2x - 5)(x + 4)$.
9. $(2r^2 - 7)(3r^2 + 5)$.
10. $(p^2 - 3q)(p^2 + 7q)$.
11. $(a + \frac{1}{2})(a - \frac{1}{3})$.
12. $(\frac{1}{2}x + 5y)(\frac{1}{2}x - 5y)$.
13. $(u - \frac{1}{2})(u - \frac{1}{4})$.
14. $(2x + 3)(\frac{1}{2}x + 1)$.
15. $(3x^2 + 4bc)(3x^2 - 4bc)$.
16. $(y - 8)(y + 5)$.
17. $(x - \frac{3}{4})(x - \frac{3}{8})$.
18. $(1 - 6s)(3 + 2s)$.
19. $(2t - 7w^2)(3t - 4w^2)$.
20. $(\frac{2}{5}u - \frac{1}{8})(\frac{2}{5}u + \frac{1}{8})$.
21. $(3r - 7t)(5r + 2t)$.
22. $(11x^2 - 1)(12x^2 + 1)$.
23. $(z^2 - 6)(z^2 + 12)$.
24. $(x + 3y^2)(x - 2y^2)$.
25. $(5m^3 - 6s^2)(5m^3 + s^2)$
26. $(5x + \frac{1}{3})(5x - \frac{1}{3})$.
27. $(3x + 7)(x - 5)$.
28. $(4a - 3b^3)^2$.
29. $(2 - 3st)(5 + 2st)$.
30. $(a^2b + 6c)(a^2b - 13c)$.
31. $(lp + 5)(lp - 4)$.
32. $(a^3 + 7)(a^3 - 11)$.
33. $(3a + 5)(7a - 8)$.
34. $(1 + 8n)(1 - 9n)$.
35. $(2a - b^4)(2a + 3b^4)$.
36. $(12x - \frac{1}{3})(9x - \frac{1}{2})$.
37. $(20 - 16z)(3 + 2z)$.
38. $(r^3 + 16s)(r^3 - s)$.
39. $(a - 6x^2)(a + x^2)$.
40. $(4r + uv)(4r - 5uv)$.
41. $(6x^2 - \frac{1}{3})^2$.
42. $(1 + 23n)(5 - n)$.
43. $(x^4 - y^4)(x^4 + y^4)$.
44. $(5a^2 - 4b)(6a^2 - 5b)$.
45. $(x^2y + y^2x)(x^2y - y^2x)$.
46. $(\frac{1}{2}a + 10)(2a + \frac{1}{5})$.
47. $(9r + 2s)(3r - 4s)$.
48. $(12x^2 + 5)(4x^2 - 3)$.
49. $(a^2b^4 + 4x^2)^2$.
50. $(a^6 - b^6)(a^6 + b^6)$.
51. $(a + b)(a - b)(a^2 + b^2)(a^4 + b^4)$.
52. $(a + b + c)^2 = ?$
53. $(a + b)^3 = ?$
54 $(a - b)^3 = ?$
55. $(a + b)^4 = ?$
56. $(a - b)^4 = ?$
57. $(a + b + c + d)^2 = ?$
58. $5x^2 - [3y^2 + \{2x^2 - (y^2 + 3x^2) + 5y^2\} - x^2] = ?$
59. $ab - [4b^2 - (2a^2 - b^2) - \{-5a^2 + 2ab - 3b^2\}] = ?$
60. $3y^2 - [2y^2 + (9z - 2yz)] = ?$

302. Factoring. A rational and integral monomial is one that is made up of the product of two or more arithmetical or literal numbers. Thus 10, $7x$, $4abc$, $6a^2by^2$ are rational and integral, but $2a/b$, $3b\sqrt{x}$ are not.

The algebraic sum of any number of rational and integral monomials is called a rational and integral polynomial.

To factor an algebraic expression is to find two or more rational and integral expressions which will produce the given expression when multiplied together.

Next to the removal of a common monomial factor from all of the terms of a polynomial, as, for example, $na + nb + nc = n(a + b + c)$, the most fundamental cases of factoring are those depending upon the special products of the preceding section. Thus,

(1) *The difference of two squares equals the product of the sum and the difference of their square roots:*

$$x^2 - y^2 = (x - y)(x + y)$$

Thus

$$81a^8 - b^6 = (9a^4 - b^3)(9a^4 + b^3)$$

(2) *A trinomial is a perfect square when, and only when, two of its terms are perfect squares and the remaining term is twice the product of their square roots.*

To find the square root of a trinomial perfect square, take the square roots of each of its two perfect square terms and connect them by the sign of the remaining term.

Thus, $9a^2 - 24ab + 16b^2$ is a perfect square, since $\sqrt{9a^2} = 3a$, $\sqrt{16b^2} = 4b$ and $24ab = 2(3a)(4b)$.

Also $9a^2 + 30a + 16$ is not a perfect square, for $30a$ does not equal $2(3a)(4)$.

(3) *Trinomials of the form $x^2 + px + q$ can be factored when two numbers can be found whose product is q and whose sum is p.*

Thus $x^2 - 4x - 77 = (x + 7)(x - 11)$, for $7(-11) = -77$ and $(+7) + (-11) = -4$.

(4) *Trinomials of the form $ax^2 + bx + c$, if factorable, may be factored in accordance with the properties of the special product* (4), §**301**. In the product

$$\begin{array}{r} ax + b \\ cx + d \\ \hline acx^2 + (bc + ad)x + bd \end{array}$$

the terms acx^2 and bd are called *end products* and bcx and adx are called *cross products.* This most important case of factoring is best learned from the consideration of actual examples.

Factor $21x^2 + 5x - 4$.

From the term $21x^2$, consider as possible first terms $7x$ and $3x$, thus $(7x\quad)(3x\quad)$. For factors of (-4), try 2 and 2, with unlike signs, and signs so arranged that the cross product with larger absolute value shall be positive; thus $(7x - 2)(3x + 2)$. This gives middle term $8x$; incorrect. For (-4) try 4 and 1, with signs selected as before; thus, $(7x - 1)(3x + 4)$. Middle term $25x$; incorrect. Try $(7x + 4)(3x - 1)$. Middle term $5x$; correct.

(5) The difference of two cubes: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

Thus
$$27x^3 - y^6 = (3x)^3 - (y^2)^3$$
$$= (3x - y^2)(9x^2 + 3xy^2 + y^4)$$

(6) The sum of two cubes: $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

Thus
$$125a^3 + b^9 = (5a)^3 + (b^3)^3$$
$$= (5a + b^3)(25a^2 - 5ab^3 + b^6)$$

303. To factor a polynomial completely, first remove any monomial factor present; then factor the resulting expression by any of the type forms which apply, until prime factors have been obtained throughout. Thus,

(a) $5a^6 - 5b^6 = 5(a^6 - b^6) = 5(a^3 - b^3)(a^3 + b^3)$
$$= 5(a - b)(a^2 + ab + b^2)(a + b)(a^2 - ab + b^2)$$

(b) $42ax^2 + 10ax - 8a = 2a(21x^2 + 5x - 4)$
$$= 2a(7x + 4)(3x - 1)$$

Exercises

Factor the following expressions:

1. $\frac{1}{16}u^6 - \frac{9}{25}v^4$.
2. $9x^8 - 4y^6$.
3. $25x^4 - 1$.
4. $81 - 4x^2$.
5. $1 - 64a^2b^4c^6$.
6. $x^6 - y^8$.
7. $225 - a^6$.
8. $121x^2 - 144y^2$.
9. $49m^4 - 36x^2y^2z^2$.
10. $169 - \frac{9}{49}a^4x^2$.
11. $4x^2 - 20x + 25$.
12. $9a^2 + 6ab + b^2$.
13. $a^2b^2 - 17abc - 60c^2$.
14. $r^4 - 11r^2 + 30$.
15. $16b^2 + 30b + 9$.
16. $81u^2 + 180uv + 100v^2$.
17. $36a^2 - 132a + 121$.
18. $x^2y^4 - 4xy^2 + 4$.
19. $a^2b^2 - 2ab - 35$.
20. $u^6 + u^3 - 110$.
21. $a^4b^2 - 14a^2b + 49$.
22. $x^2 + 6x - 27$.
23. $c^3 - 64t^3$.
24. $8x^3 - 1$.
25. $1 - 13t - 68t^2$.
26. $x^4 - 6x^2b - 55b^2$.
27. $au^2 - 4auv - 45av^2$.
28. $28a^2 - a - 2$.
29. $3s^2 - 17st + 24t^2$.
30. $15r^2 - r - 6$.
31. $4y^2 - 3y - 7$.
32. $64u^6 - 27x^3$.
33. $6ar - 3as + 4at$.
34. $a^2 + 2a - 35$.
35. $9x^2 + 12xy - 32y^2$.
36. $a^2 + 10ab + 25b^2$.
37. $625x^2y^2 - \frac{1}{49}$.
38. $3cdy^2 - 9cdy - 30cd$.
39. $4ax^2 - 25ay^4$.
40. $3y^3 + 24$.
41. $4x^2 - 27x + 45$.
42. $6x^2 + 7x - 3$.

43. $\frac{9}{25}z^2 - 1.$

44. $10x^3y - 5x^2y^2 - 5xy^3.$

45. $m^2n^2 + 7mn - 30.$

46. $x^2 - 3xy - 70y^2.$

47. $mx^2 + 7mx - 44m.$

48. $x^3 - 3x^2 - 108x.$

49. $x^8 - y^8.$

50. $x^4 - 5x^2y - 24y^2.$

51. $8n^2 + 18n - 5.$

52. $3x^4 - 12.$

53. $9m^2 - 42mt + 49t^2.$

54. $10x^2 - 39x + 14.$

55. $12x^2 + 11x + 2.$

56. $36x^2 + 12x - 35.$

57. $x^3 - 8y^3.$

58. $2am^2 - 50a.$

59. $72 + 7x - 49x^2.$

60. $31x^2 + 23xy - 8y^2.$

61. $24a^2 + 26a - 5.$

62. $1 - 3xy - 108x^2y^2.$

63. $x^2 - 14mx + 40m^2.$

64. $2b + 10ab - 28a^2b.$

65. $c^3 + 27d^3.$

66. $3x^3y - 27xy^3.$

67. $\frac{1}{10}x^3y^2 - \frac{1}{40}x^5y^4.$

68. $49n^4y - 196n^2y^3.$

69. $x^2 - 16x + 48.$

70. $x^2 + 23x - 50.$

71. $a^4n^4 + 31a^2n^2 + 30.$

72. $9x^2 + 37xy + 4y^2.$

304. General Distributive Law in Multiplication. From the meaning of a product, we may write

$$\begin{aligned}(a+b+c+\dots)(x+y+z+\dots) = {} & ax+bx+cx+\dots \\ & +ay+by+cy+\dots \\ & +az+bz+cz+\dots, \\ & \text{etc.}\end{aligned}$$

Stating this in words: *The product of one polynomial by another is the sum of all the terms found by multiplying each term of one polynomial by each term of the other polynomial.*

To multiply several polynomials together, we continue the above process. In words we may state the generalized distributive law of the product of any number of polynomials as follows:

The product of k polynomials is the aggregate of ALL *of the possible partial products which can be made by multiplying together k terms, of which one and only one must be taken from each polynomial.*

Thus,

$$\begin{aligned}&(a+b+c+\dots)(x+y+z+\dots)(u+v+w+\dots) \\ &= axu + axv + \dots + ayu + ayv + \dots + azu + azv + \dots \\ &+ bxu + bxv + \dots + byu + byv + \dots + bzu + bzv + \dots \\ &+ cxu + cxv + \dots \\ &+ \dots, \text{etc.}\end{aligned}$$

If the number of terms in the different polynomials be n, r, s, t . . . respectively, the total number of terms in the product will be nrst . . . The student may prove this.

305. The Fundamental Theorem in the Factoring of $x^n \pm a^n$. *The expression* $(x^n - a^n)$ *is always divisible by* $(x - a)$.

Write
$$x^n - a^n = x^n - ax^{n-1} + ax^{n-1} - a^n$$
$$= x^{n-1}(x - a) + a(x^{n-1} - a^{n-1})$$

Now *if* $(x^{n-1} - a^{n-1})$ is divisible by $(x - a)$, then plainly $x^{n-1}(x - a) + a(x^{n-1} - a^{n-1})$ is also divisible by $(x - a)$. But this last expression equals $(x^n - a^n)$, as we have shown. Therefore, *if* $(x - a)$ exactly divides $(x^{n-1} - a^{n-1})$, it will also exactly divide $(x^n - a^n)$.

But $(x - a)$ *will* exactly divide $(x^3 - a^3)$, therefore it will divide $(x^4 - a^4)$, and since $(x - a)$ exactly divides $(x^4 - a^4)$ it will exactly divide $(x^5 - a^5)$, and so on.

Therefore, *whatever* positive whole number be represented by n, $(x - a)$ will exactly divide $(x^n - a^n)$.

We see that $(x - a)$ is *one* factor of $(x^n - a^n)$. The *other* factor of $(x^n - a^n)$ is found by actually dividing $(x^n - a^n)$ by $(x - a)$. Thus

$$(x^n - a^n) = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-2}x + a^{n-1})$$

The student may show that $(x + a)$ divides $x^n + a^n$ if n be odd, and divides $x^n - a^n$ if n be even.

306. Quadratic equations are usually solved (a) by factoring, (b) by completing the square, or (c) by use of a formula.

(a) To solve by factoring, transpose all terms to the left member of the equation and completely factor. The solution of the equation is then deduced from the fact that if the value of a product is zero, then one of the factors must equal zero. Thus

(1) Solve the equation

$$x^2 + 54 = 15x$$

Transposing $\quad x^2 - 15x + 54 = 0$

Factoring $\quad (x - 9)(x - 6) = 0$

$$x - 9 = 0 \text{ if } x = 9$$
$$x - 6 = 0 \text{ if } x = 6$$

Hence the roots of the equation are 9 and 6.

CHECK: Does $(9)^2 + 54 = 15 \times 9$?
Does $(6)^2 + 54 = 15 \times 6$?

(b) To solve by completing the square, use the properties of $(x \pm a)^2 = x^2 \pm 2ax + a^2$, as follows:

(2) Solve $x^2 - 12x = 13$.

Add the square of 1/2 of 12 to each side

$$x^2 - 12x + 36 = 49$$

Take the square root of each member

$$x - 6 = \pm 7$$

Hence

$$x = 6 + 7 = 13$$
$$x = 6 - 7 = -1$$

CHECK: Does $(13)^2 - 12 \times 13 = 13$?
Does $(-1)^2 - 12 \times (-1) = 13$?

Since in general $(x - a)(x - b) = x^2 - (a + b)x + ab$, we can check thus:

Does $13 + (-1) = -(-12)$?
Does $13(-1) = -13$?

(3) Solve $x^2 - 20x + 97 = 0$.

Transpose 97 and add the square of 1/2 of 20 to each side:

$$x^2 - 20x + 100 = -97 + 100 = 3$$

Take the square root of each number:

$$x - 10 = \pm \sqrt{3}$$

Hence

$$x_1 = 10 + \sqrt{3}$$
$$x_2 = 10 - \sqrt{3}$$

CHECK: Does $x_1 + x_2 = -(-20)$?
Does $x_1 x_2 = 97$?

(c) To solve by use of a formula, first solve

$$ax^2 + bx + c = 0 \tag{1}$$

The roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2}$$

For a particular example, substitute the appropriate values of a, b, and c. Thus:

(4) Solve $2x^2 - 3x - 5 = 0$.

Comparing the equation term by term with (1) we have

$$a = 2,\ b = -3,\ c = -5$$

Substitute these values in the formula (2)

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)}$$
$$= \frac{3 \pm 7}{4}$$

Therefore

$$x_1 = 5/2, \; x_2 = -1$$

CHECK: Does $x_1 + x_2 = -b/a = 3/2$?
Does $x_1 x_2 = c/a = -5/2$?

Exercises

Solve the following quadratics in any manner:

1. $x^2 + 5x + 6 = 0$.
2. $x^2 + 4x = 96$.
3. $x^2 = 110 + x$.
4. $x^2 + 5x = 0$.
5. $6x^2 + 7x + 2 = 0$.
6. $8x^2 - 10x + 3 = 0$.
7. $x^2 + mx - 2m^2 = 0$.
8. $3t^2 - t - 4 = 0$.
9. $10r^2 + 7r = 12$.
10. $x^2 + 2ax = b$.
11. $x^2 + 4x = 5$.
12. $x^2 + 6x = 16$.
13. $2x^2 - 20x = 48$.
14. $x^2 + 3x = 18$.
15. $x^2 + 5x = 36$.
16. $3x^2 + 6x = 9$.
17. $4x^2 - 4x = 8$.
18. $x^2 - 7x = -6$.
19. $x^2 - ax = 6a^2$.
20. $x^2 - 2ax = 3a^2$
21. $x^2 - x = 2$.
22. $x^2 + x = a^2 + a$.
23. $x^2 - 10x = -9$.
24. $2x^2 - 15x = 50$.
25. $x^2 + 8x = -15$.
26. $3x^2 + 12x = 36$.
27. $2x^2 + 10x = 100$.
28. $x^2 - 5x = -4$.
29. $3x^2 - 12ax = 63a^2$.
30. $4x^2 - 12ax = 16a^2$.
31. $x^2 - x = 6$.
32. $x^2 + 7x = -12$.
33. $x^2 - 5x = 14$.
34. $x^2 + x = 12$.
35. $x^2 - x = 12$.
36. $x^2 = 6x - 5$.
37. $x^2 = -4x + 21$.
38. $x^2 = -4x + 5$.
39. $x^2 + 5x + 6 = 0$.
40. $x^2 + 11x = -30$.
41. $x^2 - 7x + 12 = 0$.
42. $x^2 - 13x = 30$.
43. $3x^2 + 4x = 7$
44. $3x^2 + 6x = 24$.
45. $4x^2 - 5x = 26$.
46. $5x^2 - 7x = 24$.
47. $2x^2 - 35 = 3x$.
48. $3x^2 - 50 = 5x$.
49. $3x^2 - 24 = 6x$.
50. $2x^2 - 3x = 104$.
51. $2x^2 + 10x = 300$.
52. $3x^2 - 10x = 200$.
53. $4x^2 - 7x + \frac{1}{2} = 0$.
54. $\frac{2}{3}x^2 - \frac{3}{2}x = -\frac{19}{27}$.
55. $9x^2 + 6x - 43 = 0$.
56. $18x^2 - 3x - 66 = 0$.

57. $\frac{1}{2}x^2 - 3x + \frac{17}{18} = 0.$

58. $\frac{x^2}{4} - \frac{3x}{2} + 2 = 0.$

59. $2x^2 - 22x = -60.$

60. $3x^2 + 7x - 370 = 0.$

61. $5x^2 - \frac{1}{2}x - \frac{7}{18} = 0.$

62. $\frac{x^2}{3} - \frac{x}{2} + \frac{1}{6} = 0.$

63. $x^2 + 2x + 1 = 6x + 6.$

64. $x^2 - 49 = 10(x - 7).$

65. $2x^2 + 60x = -400.$

66. $a^2 + 7a + 7 = 0.$

67. $z^2 = 3z + 2.$

68. $r = r^2 - 3.$

69. $s^2 = 5s + 6.$

70. $r^2 + 3r = 4.$

71. $2s^2 + 4as - c = 0.$

72. $x^2 + 6ax - 5 = 0.$

73. $x^2 - 10ax = -9a^2.$

74. $cx^2 + 2dx + e = 0.$

75. $2x^2 + 6x - n = 0.$

76. $y^2 + \frac{2}{5}y = \frac{3}{5}.$

77. $x^2 = 5 + \frac{4}{3}x.$

78. $u^2 - \frac{6}{7}u - 1 = 0.$

79. $t^2 + \frac{1}{5}t = \frac{4}{5}.$

80. $r^2 - \frac{5}{2} = \frac{3}{4}r.$

81. $s^2 - \frac{5}{6}s = \frac{25}{6}.$

82. $3r^2 - 2r = 40.$

83. $4x^2 - 3x = 3.$

84. $9t^2 + 4t = 6.$

85. $5(x^2 - 25) = x - 5.$

86. $9u^2 + 18u + 8 = 0$

87. $x^2 + px + q.$

88. $x^4 - 8x^2 + 15 = 0.$

89. $u^4 - 29u^2 + 100 = 0.$

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90. $\frac{2}{x^2} = 1 - \frac{8}{3x}.$

91. $2y + \frac{5}{2} = \frac{5}{4y}.$

92. $\frac{1}{5} + \frac{3}{4x} = \frac{5}{4x^2}.$

93. $\frac{5}{5 - x} + \frac{8}{8 - x} = 3.$

94. $\frac{n - 3}{n - 2} - \frac{n + 4}{n} = \frac{3}{2}.$

95. $3x + \frac{7x + 11}{x} = 2 + \frac{33}{x}.$

96. $\frac{24}{x} - \frac{24}{x - 2} + 1 = 0.$

97. $x^6 - 35x^3 + 216 = 0.$

98. $x^2 + \frac{100}{x^2} = 29.$

99. $\left(x + \frac{1}{x}\right)^2 - \frac{16}{3}\left(x + \frac{1}{x}\right) + 7 = \frac{1}{3}.$

100. $u + \frac{1}{u} = a + \frac{1}{a}.$

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307. The Definitions of Exponents.

(1) n a positive integer: $a^n = aaa \ldots$ to n factors.

(2) n and r positive integers: $a^{1/r} = \sqrt[r]{a}$ and $a^{n/r} = (\sqrt[r]{a})^n = \sqrt[r]{a^n}$.

(3) $a^0 = 1$.

(4) n any number, positive or negative, integral or fractional: $a^{-n} = 1/a^n$.

308. The Laws of Exponents. For n and r any numbers, positive or negative, integral or fractional:

(1) $a^n a^r = a^{n+r}$, or law for multiplication and division.

(2) $(a^n)^r = a^{nr}$, or law for involution.
(3) $a^n b^n = (ab)^n$, or distributive law of exponents.

NOTE: The student must distinguish between $-a^n$ and $(-a)^n$. Thus $-8^{1/3} = -2$, and $(-8)^{1/3} = -2$, but $(-3)^2 = 9$ and $-3^2 = -9$.

Exercises 1

Use the definitions of exponents (1), (2), (3), (4) §**307**, and the laws of exponents (1), (2), (3), §**308**, and find the results of the indicated operations in the following exercises.

1. $x^{12}x^{13}$.
2. $a^{2n}a^{3n}$.
3. $x^{2n+1}x^n$.
4. b^2b^{n+5}.
5. $m^{n+1}m^{n-1}$.
6. $a^{n-2}a^{3+n}$.
7. $x^{n-r+1}x^r$.
8. $m^{5a}m^{-2a}$.
9. $x^{13} \div x^8$.
10. $x^8 \div x^{13}$.
11. $a^{3n} \div a^n$.
12. $e^{n+5} \div e^3$.
13. $10^{2r+3} \div 10^r$.
14. $n^{r+6} \div n^{r+3}$.
15. $u^{n+r} \div u^{n-r}$.
16. $x^{n-r+1} \div x^r$.
17. $(a^7)^3$.
18. $(a^4)^6$.
19. $(-ab^2)^3$.
20. $(a^2y^2)^5$.
21. $(b^m)^2$.
22. $(-a^nb^r)^3$.
23. $(a^3b^5)^t$.
24. $(r^ms^n)^p$.
25. $\left(\frac{x^5}{y^6}\right)^4$.
26. $\left(\frac{a^3}{b^7}\right)^n$.
27. $\left(-\frac{x^n}{y^r}\right)^s$.

Exercises 2

Write each of the following sixteen expressions, using fractional exponents in place of radical signs:

1. $\sqrt{a}$.
2. $\sqrt{a^3}$.
3. $\sqrt[3]{a^2}$.
4. $\sqrt[5]{a^3}$.
5. $\sqrt[7]{a^5}$.
6. $(\sqrt{a})^6$.
7. $\sqrt[7]{a^5}$.
8. $(\sqrt[4]{a})^5$.
9. $\sqrt[8]{x^3}$.
10. $(\sqrt[8]{x})^3$.
11. $\sqrt[8]{x^t}$.
12. $(\sqrt[8]{x})^t$.
13. $\sqrt[5]{a-5}$.
14. $(\sqrt[8]{a-b})^t$.
15. $\sqrt{a^2-b^2}$.
16. $\sqrt[8]{(a+b)^t}$.

Find the numerical value of each of the following sixteen expressions:

17. $4^{\frac{1}{2}}$.
18. $27^{\frac{1}{3}}$.
19. $9^{\frac{1}{2}}$.
20. $16^{\frac{1}{4}}$.
21. $625^{\frac{1}{4}}$.
22. $64^{\frac{1}{6}}$.
23. $216^{\frac{1}{3}}$.
24. $16^{\frac{5}{4}}$.
25. $81^{\frac{3}{2}}$.
26. $125^{\frac{5}{3}}$.
27. $32^{\frac{4}{5}}$.
28. $81^{\frac{5}{4}}$.
29. $256^{\frac{3}{4}}$.
30. $64^{\frac{5}{6}}$.
31. $512^{\frac{2}{3}}$.
32. $128^{\frac{4}{7}}$.

Write each of the following expressions *in two ways*, using radical signs instead of fractional exponents:

33. $a^{\frac{1}{3}}$.
34. $1^{\frac{2}{5}}$.
35. $m^{\frac{3}{4}}$.
36. $x^{\frac{3}{7}}$.
37. $n^{\frac{2}{3}}$.
38. $b^{\frac{3}{4}}$.
39. $e^{\frac{5}{6}}$.
40. $h^{\frac{1}{9}}$.
41. $r^{\frac{7}{8}}$.
42. $x^{\frac{p}{q}}$.
43. $y^{\frac{1}{t}}$.
44. $a^{\frac{x}{y}}$.
45. $a^{\frac{n}{3}}$.
46. $b^{\frac{r}{2n}}$.
47. $x^{\frac{n+1}{r}}$.
48. $a^{\frac{r+s}{t}}$.

Exercises 3

Perform the indicated operations in each of the following examples by means of the laws of exponents.

1. $a^{\frac{2}{3}} \times a^{\frac{3}{4}}$.

$$a^{\frac{2}{3}} \times a^{\frac{3}{4}} = a^{\frac{2}{3}+\frac{3}{4}} = a^{\frac{8}{12}+\frac{9}{12}} = a^{\frac{17}{12}}.$$

2. $x^{\frac{2}{3}} \times x^{\frac{3}{2}}$.
3. $x^{\frac{1}{5}} \times x^{\frac{3}{10}}$.
4. $x^{\frac{1}{4}} \times x^{\frac{1}{5}}$.
5. $a^{\frac{3}{7}} \times a^{\frac{2}{3}}$.
6. $x^{\frac{1}{2n}} \times a^{\frac{4}{3n}}$
7. $a^{\frac{2}{n}} \times a^{\frac{r}{2n}}$.

8. $a^{\frac{5}{7}} \div a^{\frac{2}{3}}$.

$$a^{\frac{5}{7}} \div a^{\frac{2}{3}} = a^{\frac{5}{7}-\frac{2}{3}} = a^{\frac{15}{21}-\frac{14}{21}} = a^{\frac{1}{21}}.$$

9. $h^{\frac{2}{3}} \div h^{\frac{1}{3}}$.
10. $m^{\frac{3}{5}} \div m^{\frac{3}{10}}$.
11. $8a^5b^{\frac{3}{2}} \div 4a^2b^{\frac{1}{2}}$.
12. $9a^{\frac{3}{5}} \div a^{\frac{1}{2}}$.
13. $6a^{\frac{2}{3}} \div 3a^{\frac{1}{2}}$.
14. $ab^{\frac{n}{r}} \div a^{\frac{1}{2}}\, b^{\frac{n}{4r}}$.

15. $(a^{\frac{4}{5}})^{\frac{7}{8}}$.

$$(a^{\frac{4}{5}})^{\frac{7}{8}} = a^{\frac{28}{40}} = a^{\frac{7}{10}}.$$

16. $(a^{\frac{2}{3}})^{\frac{6}{10}}$.
17. $(h^{\frac{2}{5}})^{\frac{5}{6}}$.
18. $(a^{\frac{3}{5}})^{\frac{5}{6}}$.
19. $(a^{\frac{5}{12}})^4$.
20. $[(x^n)^{\frac{1}{n}}]^{\frac{n}{r}}$.
21. $(x^{\frac{st}{6rn}})^{\frac{2}{t}}$.

22. $(a^{\frac{2}{3}}x^{\frac{1}{2}}y^{\frac{3}{4}})^{\frac{6}{5}}$.

$$(a^{\frac{2}{3}}x^{\frac{1}{2}}y^{\frac{3}{4}})^{\frac{6}{5}} = (a^{\frac{2}{3}})^{\frac{6}{5}}(x^{\frac{1}{2}})^{\frac{6}{5}}(y^{\frac{3}{4}})^{\frac{6}{5}} = a^{\frac{4}{5}}x^{\frac{3}{5}}y^{\frac{9}{10}}.$$

23. $(a^2b^{\frac{1}{2}})^{\frac{1}{2}}$.
24. $(ad^{\frac{1}{2}})^{\frac{2}{5}}$.
25. $(36a^4x^2y^3)^{\frac{1}{2}}$.
26. $(a^{\frac{1}{2}}x^{\frac{1}{7}}y^{\frac{1}{8}})^{14}$.
27. $(32x^{\frac{3}{8}}y^{\frac{9}{4}})^{\frac{4}{9}}$.
28. $(\frac{1}{8}a^5b^3c)^{\frac{1}{8}}$.

29. $\left(\dfrac{a^{\frac{2}{3}}}{b^{\frac{3}{4}}}\right)^{\frac{1}{6}}$

$$\left(\frac{a^{\frac{2}{3}}}{b^{\frac{3}{4}}}\right)^{\frac{1}{6}} = \frac{(a^{\frac{2}{3}})^{\frac{1}{6}}}{(b^{\frac{3}{4}})^{\frac{1}{6}}} = \frac{a^{\frac{1}{9}}}{b^{\frac{1}{8}}}.$$

30. $\left(\dfrac{x}{y^{\frac{1}{2}}}\right)^{\frac{1}{2}}$.
31. $\left(\dfrac{a^2}{b^3}\right)^{\frac{1}{5}}$.
32. $\left(\dfrac{a^3}{b^{\frac{2}{3}}}\right)^{\frac{2}{7}}$.
33. $\left(\dfrac{a^5}{b^{\frac{2}{3}}}\right)^{\frac{3}{5}}$.
34. $\left(\dfrac{8x^{\frac{1}{2}}}{y^{\frac{2}{3}}}\right)^{\frac{2}{3}}$.
35. $\left(\dfrac{9a^{\frac{3}{4}}}{4b^{\frac{2}{5}}}\right)^{\frac{3}{2}}$.

36. $(a^{\frac{4}{3}} + a^{\frac{2}{3}} + 1)(a^{\frac{5}{3}} + a - a^{\frac{1}{3}})$.

We arrange the work thus:

$$\begin{array}{llll}
a^{\frac{4}{3}} + a^{\frac{2}{3}} + 1 & & & \\
a^{\frac{5}{3}} + a \;\; - a^{\frac{1}{3}} & & & \\
\hline
a^{3} + a^{\frac{7}{3}} + a^{\frac{5}{3}} & & & \\
\quad\;\; a^{\frac{7}{3}} + a^{\frac{5}{3}} + a & & & \\
\qquad\qquad - a^{\frac{5}{3}} - a - a^{\frac{1}{3}} & & & \\
\hline
a^{3} + 2a^{\frac{7}{3}} + a^{\frac{5}{3}} \qquad - a^{\frac{1}{3}} & & &
\end{array}$$

37. $(x + 2y^{\frac{1}{2}} + 3y^{\frac{1}{3}})(x - 2y^{\frac{1}{2}} + 3y^{\frac{1}{3}})$.
38. $(x^{\frac{3}{4}} + y^{\frac{3}{4}})(x^{\frac{3}{4}} - y^{\frac{3}{4}})$.
39. $(a^{\frac{4}{5}} - 3a^{\frac{3}{5}}b^{\frac{1}{2}} + 4a^{\frac{2}{5}}b - a^{\frac{1}{5}}b^{\frac{3}{2}})(a^{\frac{2}{5}} - 2a^{\frac{1}{5}}b^{\frac{1}{2}})$.
40. $(a^{\frac{3}{n}} - 2a^{\frac{2}{n}} + 3a^{\frac{1}{n}})(2a^{\frac{2}{n}} - a^{\frac{1}{n}})$.

Exercises 4

Find the numerical value of each of the following:

1. 2^{-1}.
2. 4^{-2}.
3. $(-2)^{-3}$.
4. 10^{-5}.
5. 1^{-1}
6. 2^{-2}.
7. 2^{-4}.
8. $16^{-\frac{3}{4}}$.
9. $81^{-\frac{1}{2}}$.
10. $1024^{-\frac{3}{5}}$.
11. $512^{-\frac{1}{3}}$.
12. $625^{-\frac{3}{4}}$.
13. $\frac{1}{2^{-1}}$.
14. $\frac{2}{3^{-2}}$.
15. $\frac{5}{(-4)^{-3}}$.
16. $\frac{1^{-8}}{8^{-1}}$.
17. $\frac{5^{-2}}{36^{-\frac{1}{2}}}$.
18. $\frac{32^{-\frac{1}{5}}}{2^{-1}}$.
19. $\frac{16^{-\frac{3}{4}}}{5^{-2}}$.
20. $\frac{7^{-1}}{49^{-\frac{1}{2}}}$.

Write each of the following expressions without using negative exponents:

21. x^{-2}.
22. x^2y^{-2}.
23. $\frac{1}{x^{-2}}$.
24. $\frac{m^{-3}}{x^{-6}}$.
25. $5a^{-5}$.
26. $3a^{-2}b^{-\frac{1}{2}}$.
27. $\frac{2a^{-2}}{3b^2x^{-3}}$.
28. $\frac{a^2b^{-4}}{x^{-3}y^{-5}}$.
29. $(x + y)^{-2}$.
30. $(-x)^{-3}$.
31. $\frac{x^4}{y^{-5}}$.
32. $\frac{3a^{\frac{2}{3}}b^{-1}}{5a^{-\frac{2}{3}}b}$.
33. $2a^3x^{-2}y^{-\frac{1}{2}}$.
34. $(-a^2)^{-3}$.
35. $\frac{a^{-\frac{2}{3}}b^{\frac{1}{6}}}{3x^{-\frac{1}{2}}b^{-\frac{1}{6}}}$.
36. $\frac{3a^2b^{-2}c^{-4}}{5a^{-1}b^{-3}c^{-5}}$.

Write each of the following expressions *in one line:*

37. $\frac{1}{a^2}$.
38. $\frac{4}{a^r}$.
39. $\frac{ab^2}{c^3d^4}$.
40. $\frac{2x^{-1}y^5}{3a^{-2}y^{-3}}$.
41. $\frac{3x^3y^{-3}}{4r^{-2}t^5}$.
42. $\frac{4x^{-5}y^{-6}}{u^4z^{-7}}$.
43. $\frac{x^{-4}y^2}{5a^nb^{-r}}$.
44. $\frac{ab^{\frac{1}{2}}}{x^{-\frac{1}{3}}y^{s/t}}$.
45. $\frac{16(a+b)^{-1}c^{\frac{1}{2}}}{(a-b)^{-\frac{2}{3}}c^{-\frac{5}{2}}}$.
46. $\frac{a}{x^3} + \frac{b}{x^2} + \frac{c}{x} + \frac{d}{x^{-1}}$.

Exercises 5

Perform the indicated operations in each of the following by means of the laws of exponents.

1. $a^8 \times a^{-5}$.
2. $r^{12} \times r^{-10}$.
3. $c^{-8} \div c^{-5}$.
4. $8a^{-4} \times 3a^2$.
5. $u^{-1} \times u^{\frac{3}{4}}$.
6. $x^5 \div x^{-11}$.
7. $m^{-\frac{2}{3}} \times m^{-\frac{1}{3}}$.
8. $6ax^{-6} \times \frac{1}{2}bx^2$.
9. $a^{-3}b^{-n} \div ab^{-r}$.
10. $(-7a^{-3}b^{-2})(-4a^2b^{-1})(a^{-2}b^2x^{-1})$.
11. $(2a^{\frac{1}{2}}b^{-\frac{2}{3}})(a^{-\frac{3}{2}}b^{\frac{2}{3}} - \frac{1}{2}a^{\frac{2}{3}}b^{\frac{3}{2}} + a^{\frac{3}{2}}b^{-\frac{2}{5}})$.
12. $7a^{-1}b^{-2}c^{-3} \div 8a^{-2}b^{-3}c^{-4}$.
13. $56x^5y^{-7}z^4 \div 7x^{-1}y^{-8}z^{-4}$.
14. $18a^{-\frac{1}{2}}b^{\frac{2}{3}}c^{-5} \div 6a^{\frac{3}{2}}b^{\frac{1}{3}}c^{-5}$.
15. $6x^{\frac{1}{3}}y^{-\frac{2}{3}}z^{\frac{1}{6}} \div 2x^{-\frac{2}{3}}y^{\frac{1}{3}}z^{-\frac{1}{2}}$.
16. $(a^{-3})^2$.
17. $(a^{-2})^{-5}$.
18 $(a^5)^{-2}$.
19. $(n^{\frac{1}{3}})^{-3}$.
20. $(r^{-\frac{3}{4}})^{-\frac{2}{3}}$.
21. $(c^{-8})^{\frac{3}{4}}$.
22. $(abc)^{-4}$.
23. $(a^{-4}b^2c^6)^{-3}$.
24. $(x^{\frac{5}{6}}y^{\frac{3}{4}})^{-12}$.
25. $(8x^3y^{-6})^{-\frac{1}{3}}$.
26. $(x^{-\frac{3}{4}}b^{\frac{5}{6}})^{\frac{10}{3}}$.
27. $(a^{-5}b^{-10})^{-\frac{2}{5}}$.
28. $(-1x^3)^{-4}$.
29. $(-a^{-4})^{-3}$.
30. $(-a^6)^{-\frac{4}{3}}$.
31. $(-8^{-3}b^{\frac{3}{4}}c^{-\frac{1}{3}})^{-\frac{1}{3}}$.
32. $(-1x^3y^{-9})^{\frac{1}{3}}$.
33. $\left(\begin{array}{c} a \\ \end{array}\right)^{-5}$.
34. $\left(\frac{a^{-5}}{b^{-4}}\right)^{-7}$.
35. $\left(\frac{a^{-4}}{b^{-6}}\right)^{-2}$.
36. $\left(\frac{x^3}{y^9}\right)^{-\frac{1}{2}}$.
37. $\left(\frac{a^{-2}b^3}{x^{-1}y^{-4}}\right)^2$.
38. $\left(\frac{a^{-3}b}{x^3y^{-2}}\right)^{-3}$.
39. $\left(\frac{a^2b^{-2}}{xy^3}\right)^3$.
40. $\left(\frac{a^5b^{-3}}{3x^{-1}y^2}\right)^{-1}$.
41. $\left(\frac{a^6b^{-3}}{x^{-9}y^{-\frac{3}{4}}}\right)^{-\frac{1}{3}}$.
42. $(a^2x^{-1}+3a^3x^{-2})(4a^{-1} - 5x^{-1} + 6ax^{-2})$

$$
\begin{array}{rrrr}
4a^{-1} - 5x^{-1} + 6ax^{-2} & & & \\
a^2x^{-1} + 3a^3x^{-2} & & & \\
\hline
4ax^{-1} & - 5a^2x^{-2} & + 6a^3x^{-3} & \\
 & 12a^2x^{-2} & - 15a^3x^{-3} & + 18a^4x^{-4} \\
\hline
4ax^{-1} & + 7a^2x^{-2} & - 9a^3x^{-3} & + 18a^4x^{-4}
\end{array}
$$

43. $(2x^{-\frac{1}{5}} - 3x + 4x^{\frac{1}{5}})(x^{-\frac{3}{5}} - 2x^{-\frac{2}{5}} + 3x^{-\frac{1}{5}})$.
44. $(x^{-\frac{4}{3}} - 2x^{-\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}})(x^{-\frac{2}{3}} - y^{\frac{2}{3}})$.
45. $(3x^{\frac{2}{3}} - \frac{3}{2}x^{\frac{1}{3}} + 4) \times 2x^{-\frac{1}{3}}$.
46. $(x^{-\frac{2}{3}} + x^{-\frac{1}{3}} + 1)(x^{-\frac{1}{3}} - 1)$.
47. $(x^{-\frac{1}{2}} + y^{-2})(x^{-\frac{1}{2}} - y^{-2})$.
48. $(x^{\frac{1}{4}}y + y^{\frac{3}{3}})(x^{\frac{1}{4}} - y^{-\frac{1}{3}})$.

49. $(2a^{\frac{3}{2}} - 3ax^{\frac{1}{2}})(3a^{-\frac{1}{2}} + 2x^{-\frac{1}{2}})(4a^{\frac{1}{2}}x^{\frac{1}{2}} + 9a^{-\frac{1}{2}}x^{\frac{3}{2}})$.

50. $(x^{-\frac{3}{2}} - x^{-1}y^{\frac{1}{2}} + x^{-\frac{1}{2}}y - y^{\frac{3}{2}}) \div (x^{-\frac{1}{2}} - y^{\frac{1}{2}})$.

$$\begin{array}{l} x^{-\frac{1}{2}} - y^{\frac{1}{2}})\,x^{-\frac{3}{2}} - x^{-1}y^{\frac{1}{2}} + x^{-\frac{1}{2}}y - y^{\frac{3}{2}}\,(x^{-1} + y \\ \qquad\qquad \underline{x^{-\frac{3}{2}} - x^{-1}y^{\frac{1}{2}}\qquad\qquad\qquad} \\ \qquad\qquad\qquad\qquad\qquad x^{-\frac{1}{2}}y - y^{\frac{3}{2}} \\ \qquad\qquad\qquad\qquad\qquad \underline{x^{-\frac{1}{2}}y - y^{\frac{3}{2}}} \end{array}$$

51. $(x^{-3} + 2x^{-2} - 3x^{-1}) \div (x^{-2} + 3x^{-1})$.

309. Reduction of Surds or Radicals.

1\. *If any factor of the number under the radical sign is an exact power of the indicated root, the root of that factor may be extracted and written as the coefficient of the surd, while the other factors are left under the radical sign.*

(1) Thus,
$$\begin{aligned} \sqrt{8} &= \sqrt{4 \times 2} \\ &= \sqrt{4}\sqrt{2} \\ &= 2\sqrt{2} \end{aligned}$$

(2) Also,
$$\begin{aligned} \sqrt[3]{81} &= \sqrt[3]{27 \times 3} \\ &= \sqrt[3]{27}\sqrt[3]{3} \\ &= 3\sqrt[3]{3} \end{aligned}$$

(3) Also,
$$\begin{aligned} \sqrt[3]{16ax^4} &= \sqrt[3]{8x^3 \times 2ax} \\ &= \sqrt[3]{3x^3}\sqrt[3]{2ax} \\ &= 2x\sqrt[3]{2ax} \end{aligned}$$

2\. *The expression under the radical sign of any surd can always be made integral.*

(1) Thus
$$\begin{aligned} \sqrt[3]{\frac{2}{3}} &= \sqrt[3]{\frac{2}{3} \times \frac{9}{9}} = \sqrt[3]{\frac{18}{27}} \\ &= \sqrt[3]{\frac{1}{27} \times 18} \\ &= \frac{1}{3}\sqrt[3]{18} \end{aligned}$$

(2) Also
$$\begin{aligned} \sqrt[4]{\frac{7}{8}} &= \sqrt[4]{\frac{7}{8} \times \frac{2}{2}} = \sqrt[4]{\frac{1}{16} \times 14} \\ &= \frac{1}{2}\sqrt[4]{14} \end{aligned}$$

3. *We may change the index of some surds in the following manner:*

(1) Thus,
$$\sqrt[4]{4} = \sqrt{\sqrt{4}} = \sqrt{2}$$

(2) Also,
$$\sqrt[6]{1000} = \sqrt{\sqrt[3]{1000}} = \sqrt{10}$$

(3) Also,
$$\sqrt{256c^2a^8} = \sqrt[4]{\sqrt{256c^2a^8}} = \sqrt[3]{16ca^4}$$

A surd is in its **simplest form** when (1) no factor of the expression under the radical sign is a perfect power of the required root, (2) the expression under the radical sign is integral, (3) the index of the surd is the lowest possible.

Methods of making the different reductions required by this definition have already been explained. We give a few examples.

(1) Simplify $\sqrt[9]{\dfrac{a^3}{8b^6}}$.

$$\sqrt[9]{\frac{a^3}{8b^6}} = \sqrt[3]{\frac{a}{2b^2}} = \frac{1}{2b}\sqrt[3]{4ab}$$

(2) Simplify $\sqrt[4]{\dfrac{400}{9}}$.

$$\sqrt[4]{\frac{400}{9}} = \sqrt{\frac{20}{3}} = \frac{1}{3}\sqrt{60} = \frac{2}{3}\sqrt{15}$$

(3) Simplify $\dfrac{5}{2}\sqrt[6]{\dfrac{512}{125}}$.

$$\frac{5}{2}\sqrt[6]{\frac{512}{125}} = \frac{5}{2}\sqrt{\frac{8}{5}} = 5\sqrt{\frac{2}{5}} = \sqrt{10}$$

In any piece of work it is usually expected that all the surds will finally be left in their simplest form.

Exercises

Reduce each of the following surds to its simplest form:

1. $\sqrt{\frac{9}{5}}$.
2. $\sqrt[3]{\frac{16}{3}}$.
3. $\sqrt[4]{\frac{25}{4}}$.
4. $\sqrt[6]{\frac{64}{27}}$.
5. $\sqrt[6]{\frac{64}{81}}$.
6. $\sqrt[3]{\frac{5}{12}}$.
7. $\sqrt[4]{\frac{a^2}{b^4x^6}}$.
8. $\sqrt{1-\frac{1}{x^2}}$.

9. Simplify $\sqrt{12} + \frac{1}{10}\sqrt{75} + 6\sqrt{\frac{1}{12}}$.
10. Simplify $1 + \sqrt{8} + \sqrt{2} - \sqrt{27} - \sqrt{12} + \sqrt{75}$.
11. Simplify $\sqrt[3]{21 + 7\sqrt{2}} \times \sqrt[3]{21 - 7\sqrt{2}}$.
12. Find the value of $x^2 - 6x + 7$ if $x = 3 - \sqrt{3}$.
13. Find the value when $x = \sqrt{3}$ of the expression
$$\frac{2x-1}{(x-1)^2} - \frac{2x+1}{(x+1)^2}.$$
14. Find the value of
$$(35\sqrt{10} + 77\sqrt{2} + 63\sqrt{3})(\sqrt{10} + \sqrt{2} + \sqrt{3}).$$

Solve and check each of the following equations:

15. $\sqrt{x+4} = 4$.
16. $\sqrt{2x+6} = 4$.
17. $\sqrt{10x+16} = 5$.
18. $\sqrt{2x+7} = \sqrt{5x-2}$.
19. $14 + \sqrt[3]{4x-40} = 10$.
20. $\sqrt{16x+9} = 4\sqrt{4x} - 3$.
21. $\sqrt{\frac{15}{4} + x} = \frac{3}{2} + \sqrt{x}$.
22. $\sqrt{x} - \sqrt{x-5} = \sqrt{5}$.
23. $\sqrt{x-7} = \sqrt{x-14} + 1$.
24. $\sqrt{x-7} = \sqrt{x+1} - 2$.
25. $x = 7 - \sqrt{x^2-7}$.
26. $\sqrt{x+20} - \sqrt{x-1} - 3 = 0$.
27. $\sqrt{x+3} + \sqrt{3x-2} = 7$.
28. $\sqrt{2x+1} + \sqrt{x-3} = 2\sqrt{x}$.
29. $$\frac{20x}{\sqrt{10x-9}} - \sqrt{10x-9} = \frac{18}{\sqrt{10x-9}} + 9.$$
30. $$\frac{x-1}{\sqrt{x}-1} = \frac{\sqrt{x}+1}{x-3}.$$
31. $$\frac{\sqrt{x}+\sqrt{x-3}}{\sqrt{x}-\sqrt{x-3}} = \frac{3}{x-3}.$$

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	25	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969					3	7	11	14	18	21	25	28	32
							1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553							3	6	9	12	15	18	21	24	28
					1584	1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903					3	6	9	11	14	17	20	23	26
							1931	1959	1987	2014	3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	20	23
							2455	2480	2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

°	′	log sin	d	log tan	dc	log cot	log cos			′	S	T
0	0						0.0000	0	90	0		
	10	7.4637	3011	7.4637	3011	2.5363	0.0000	50		10	6.4637	6.4637
	20	7.7648	1760	7.7648	1761	2.2352	0.0000	40		20	6.4637	6.4637
	30	7.9408	1250	7.9409	1249	2.0591	0.0000	30		30	6.4637	6.4637
	40	8.0658	969	8.0658	969	1.9342	0.0000	20		40	6.4637	6.4637
	50	8.1627	792	8.1627	792	1.8373	0.0000	10		50	6.4637	6.4638
1	0	8.2419	669	8.2419	670	1.7581	9.9999	0	89	60	6.4637	6.4638
	10	8.3088	580	8.3089	580	1.6911	9.9999	50		70	6.4637	6.4638
	20	8.3668	511	8.3669	512	1.6331	9.9999	40		80	6.4637	6.4638
	30	8.4179	458	8.4181	457	1.5819	9.9999	30		90	6.4637	6.4638
	40	8.4637	413	8.4638	415	1.5362	9.9998	20		100	6.4637	6.4638
	50	8.5050	378	8.5053	378	1.4947	9.9998	10		110	6.4637	6.4639
2	0	8.5428	348	8.5431	348	1.4569	9.9997	0	88	120	6.4636	6.4639
	10	8.5776	321	8.5779	322	1.4221	9.9997	50		130	6.4636	6.4639
	20	8.6097	300	8.6101	300	1.3899	9.9996	40		140	6.4636	6.4640
	30	8.6397	280	8.6401	281	1.3599	9.9996	30		150	6.4636	6.4640
	40	8.6677	263	8.6682	263	1.3318	9.9995	20		160	6.4636	6.4640
	50	8.6940	248	8.6945	249	1.3055	9.9995	10		170	6.4635	6.4641
3	0	8.7188	235	8.7194	235	1.2806	9.9994	0	87	180	6.4635	6.4641
	10	8.7423	222	8.7429	223	1.2571	9.9993	50		190	6.4635	6.4642
	20	8.7645	212	8.7652	213	1.2348	9.9993	40		200	6.4635	6.4642
	30	8.7857	202	8.7865	202	1.2135	9.9992	30		210	6.4635	6.4643
	40	8.8059	192	8.8067	194	1.1933	9.9991	20		220	6.4634	6.4643
	50	8.8251	185	8.8261	185	1.1739	9.9990	10		230	6.4634	6.4644
4	0	8.8436	177	8.8446	178	1.1554	9.9989	0	86	240	6.4634	6.4644
	10	8.8613	170	8.8624	171	1.1376	9.9989	50		250	6.4633	6.4645
	20	8.8783	163	8.8795	165	1.1205	9.9988	40		260	6.4633	6.4646
	30	8.8946	158	8.8960	158	1.1040	9.9987	30		270	6.4633	6.4646
	40	8.9104	152	8.9118	154	1.0882	9.9986	20		280	6.4632	6.4647
	50	8.9256	147	8.9272	148	1.0728	9.9985	10		290	6.4632	6.4648
5	0	8.9403		8.9420		1.0580	9.9983	0	85	300	6.4631	6.4649
		log cos	d	log cot	dc	log tan	log sin	′	°			

	143	142	138	137	135	134	130	129	127	125	123	122	119	117	115	114
1	14.3	14.2	13.8	13.7	13.5	13.4	13.0	12.9	12.7	12.5	12.3	12.2	11.9	11.7	11.5	11.4
2	28.6	28.4	27.6	27.4	27.0	26.8	26.0	25.8	25.4	25.0	24.6	24.4	23.8	23.4	23.0	22.8
3	42.9	42.6	41.4	41.1	40.5	40.2	39.0	38.7	38.1	37.5	36.9	36.6	35.7	35.1	34.5	34.2
4	57.2	56.8	55.2	54.8	54.0	53.6	52.0	51.6	50.8	50.0	49.2	48.8	47.6	46.8	46.0	45.6
5	71.5	71.0	69.0	68.5	67.5	67.0	65.0	64.5	63.5	62.5	61.5	61.0	59.5	58.5	57.5	57.0
6	85.8	85.2	82.8	82.2	81.0	80.4	78.0	77.4	76.2	75.0	73.8	73.2	71.4	70.2	69.0	68.4
7	100.1	99.4	96.6	95.9	94.5	93.8	91.0	90.3	88.9	87.5	86.1	85.4	83.3	81.9	80.5	79.8
8	114.4	113.6	110.4	109.6	108.0	107.2	104.0	103.2	101.6	100.0	98.4	97.6	95.2	93.6	92.0	91.2
	128.7	127.8	124.2	123.3	121.5	120.6	117.0	116.1	114.3	112.5	110.7	109.8	107.1	105.3	103.5	102.6

Formulas for using Table directly

$x < 5^\circ$: $\begin{cases} \log \sin x = \log x' + S \\ \log \tan x = \log x' + T \\ \log \cot x = \text{colog } x' + \text{co } T \end{cases}$

$x > 85^\circ$: $\begin{cases} \log \cos x = \log (90 - x)' + S \\ \log \cot x = \log (90 - x)' + T \\ \log \tan x = \text{colog } (90 - x)' + \text{co } T \end{cases}$

°	′	log sin	d	log tan	dc	log cot	log cos		
5	0	8.9403	142	8.9420	143	1.0580	9.9983	0	85
	10	8.9545	137	8.9563	138	1.0437	9.9982	50	
	20	8.9682	134	8.9701	135	1.0299	9.9981	40	
	30	8.9816	129	8.9836	130	1.0164	9.9980	30	
	40	8.9945	125	8.9966	127	1.0034	9.9979	20	
	50	9.0070	122	9.0093	123	0.9907	9.9977	10	
6	0	9.0192	119	9.0216	120	0.9784	9.9976	0	84
	10	9.0311	115	9.0336	117	0.9664	9.9975	50	
	20	9.0426	113	9.0453	114	0.9547	9.9973	40	
	30	9.0539	109	9.0567	111	0.9433	9.9972	30	
	40	9.0648	107	9.0678	108	0.9322	9.9971	20	
	50	9.0755	104	9.0786	105	0.9214	9.9969	10	
7	0	9.0859	102	9.0891	104	0.9109	9.9968	0	83
	10	9.0961	99	9.0995	101	0.9005	9.9966	50	
	20	9.1060	97	9.1096	98	0.8904	9.9964	40	
	30	9.1157	95	9.1194	97	9.8806	9.9963	30	
	40	9.1252	93	9.1291	94	0.8709	9.9961	20	
	50	9.1345	91	9.1385	93	0.8615	9.9959	10	
8	0	9.1436	89	9.1478	91	0.8522	9.9958	0	82
	10	9.1525	87	9.1569	89	0.8431	9.9956	50	
	20	9.1612	85	9.1658	87	0.8342	9.9954	40	
	30	9.1697	84	9.1745	86	0.8255	9.9952	30	
	40	9.1781	82	9.1831	84	0.8169	9.9950	20	
	50	9.1863	80	9.1915	82	0.8085	9.9948	10	
9	0	9.1943	79	9.1907	81	0.8003	9.9946	0	81
	10	9.2022	78	9.2078	80	0.7922	9.9944	50	
	20	9.2100	76	9.2158	78	0.7842	9.9942	40	
	30	9.2176	75	9.2236	77	0.7764	9.9940	30	
	40	9.2251	73	9.2313	76	0.7687	9.9938	20	
	50	9.2324	73	9.2389	74	0.7611	9.9936	10	
10	0	9.2397		9.2463		0.7537	9.9934	0	80
		log cos	d	log cot	dc	log tan	log sin	′	°

pp

	113	111	109
1	11.3	11.1	10.9
2	22.6	22.2	21.8
3	33.9	33.3	32.7
4	45.2	44.4	43.6
5	56.5	55.5	54.5
6	67.8	66.6	65.4
7	79.1	77.7	76.3
8	90.4	88.8	87.2
9	101.7	99.9	98.1

	108	107	105
1	10.8	10.7	10.5
2	21.6	21.4	21.0
3	32.4	32.1	31.5
4	43.2	42.8	42.0
5	54.0	53.5	52.5
6	64.8	64.2	63.0
7	75.6	74.9	73.5
8	86.4	85.6	84.0
9	97.2	96.3	94.5

	104	102	101
1	10.4	10.2	10.1
2	20.8	20.4	20.2
3	31.2	30.6	30.3
4	41.6	40.8	40.4
5	52.0	51.0	50.5
6	62.4	61.2	60.6
7	72.8	71.4	70.7
8	83.2	81.6	80.8
9	93.6	91.0	90.9

	99	98	97	95
1	9.9	9.8	9.7	9.5
2	19.8	19.6	19.4	19.0
3	29.7	29.4	29.1	28.5
4	39.6	39.2	38.8	38.0
5	49.5	49.0	48.5	47.5
6	59.4	58.8	58.2	57.0
7	69.3	68.6	67.9	66.5
8	79.2	78.4	77.6	76.0
9	89.1	88.2	87.3	85.5

	94	93	91	89	87	86	85	84	82	81	79	78	77	76	75	74	73
1	9.4	9.3	9.1	8.9	8.7	8.6	8.5	8.4	8.2	8.1	7.9	7.8	7.7	7.6	7.5	7.4	7.3
2	18.8	18.6	18.2	17.8	17.4	17.2	17.0	16.8	16.4	16.2	15.8	15.6	15.4	15.2	15.0	14.8	14.6
3	28.2	27.9	27.3	26.7	26.1	25.8	15.5	25.2	24.6	2.34	23.7	23.4	23.1	22.8	22.5	22.2	21.9
4	37.6	37.2	36.4	35.6	34.8	34.4	34.0	33.6	32.8	32.4	31.6	31.2	30.8	30.4	30.0	29.6	29.2
5	47.0	46.5	45.5	44.5	43.5	43.0	42.5	42.0	41.0	40.5	39.5	39.0	38.5	38.0	37.5	37.0	36.5
6	56.4	55.8	54.6	53.4	52.2	51.6	51.0	50.4	49.2	48.6	47.4	46.8	46.2	45.6	45.0	44.4	43.8
7	65.8	65.1	63.7	62.3	60.9	60.2	59.5	58.8	57.4	56.7	55.3	54.6	53.9	53.2	52.5	51.8	51.1
8	75.2	74.4	72.8	71.2	69.6	68.8	68.0	67.2	65.6	64.8	63.2	62.4	61.6	60.8	60.0	59.2	58.4
9	84.6	83.7	81.9	80.1	78.3	77.4	76.5	75.6	73.8	72.9	71.1	70.2	69.3	68.4	67.5	66.6	65.7

Formulas for using Table inversely

$$x < 5^\circ \left\{ \begin{array}{l} \log x' = \log \sin x - S \\ \log x' = \log \tan x - T \\ \operatorname{colog} x' = \log \cot x - \operatorname{co} T \end{array} \right.$$

$$x > 85^\circ \left\{ \begin{array}{l} \log (90 - x)' = \log \cos x - S \\ \log (90 - x)' = \log \cot x - T \\ \operatorname{colog} (90 - x)' = \log \tan x - \operatorname{co} T \end{array} \right.$$

°	′	log sin	d	log tan	dc	log cot	log cos	d		
10	0	9.2397	71	9.2463	73	0.7537	9.9934	3	0	80
	10	9.2468	70	9.2536	73	0.7464	9.9931	2	50	
	20	9.2538	68	9.2609	71	0.7391	9.9929	2	40	
	30	9.2606	68	9.2680	70	0.7320	9.9927	3	30	
	40	9.2674	66	9.2750	69	0.7250	9.9924	2	20	
	50	9.2740	66	9.2819	68	0.7181	9.9922	3	10	
11	0	9.2806	64	9.2887	66	0.7113	9.9919	2	0	79
	10	9.2870	64	9.2953	67	0.7047	9.9917	3	50	
	20	9.2934	63	9.3020	65	0.6980	9.9914	2	40	
	30	9.2997	61	9.3085	64	0.6915	9.9912	3	30	
	40	9.3058	61	9.3149	63	0.6851	9.9909	2	20	
	50	9.3119	60	9.3212	63	0.6788	9.9907	3	10	
12	0	9.3179	59	9.3275	61	0.6725	9.9904	3	0	78
	10	9.3238	58	9.3336	61	0.6664	9.9901	2	50	
	20	9.3296	57	9.3397	61	0.6603	9.9899	3	40	
	30	9.3353	57	9.3458	59	0.6542	9.9896	3	30	
	40	9.3410	56	9.3517	59	0.6483	9.9893	3	20	
	50	9.3466	55	9.3576	58	0.6424	9.9890	3	10	
13	0	9.3521	54	9.3634	57	0.6366	9.9887	3	0	77
	10	9.3575	54	9.3691	57	0.6309	9.9884	3	50	
	20	9.3629	53	9.3748	56	0.6252	9.9881	3	40	
	30	9.3682	52	9.3804	55	0.6196	9.9878	3	30	
	40	9.3734	52	9.3859	55	0.6141	9.9875	3	20	
	50	9.3786	51	9.3914	54	0.6086	9.9872	3	10	
14	0	9.3837	50	9.3968	53	0.6032	9.9869	3	0	76
	10	9.3887	50	9.4021	53	0.5979	9.9866	3	50	
	20	9.3937	49	9.4074	53	0.5926	9.9863	4	40	
	30	9.3986	49	9.4127	51	0.5873	9.9859	3	30	
	40	9.4035	48	9.4178	52	0.5822	9.9856	3	20	
	50	9.4083	47	9.4230	51	0.5770	9.9853	4	10	
15	0	9.4130		9.4281		0.5719	9.9849		0	75
		log cos	d	log cot	dc	log tan	log sin	d	′	°

pp

	73	71
1	7.3	7.1
2	14.6	14.2
3	21.9	21.3
4	29.2	28.4
5	36.5	35.5
6	43.8	42.6
7	51.1	49.7
8	58.4	56.8
9	65.7	63.9

	70	69
1	7.0	6.9
2	14.0	13.8
3	21.0	20.7
4	28.0	27.6
5	35.0	34.5
6	42.0	41.4
7	49.0	48.3
8	56.0	55.2
9	63.0	62.1

	68	67
1	6.8	6.7
2	13.6	13.4
3	20.4	20.1
4	27.2	26.8
5	34.0	33.5
6	40.8	40.2
7	47.6	46.9
8	54.4	53.6
9	61.2	60.3

	66	65
1	6.6	6.5
2	13.2	13.0
3	19.8	19.5
4	26.4	26.0
5	33.0	32.5
6	39.6	39.0
7	46.2	45.5
8	52.8	52.0
9	59.4	58.5

	64	63	61	60	59	58	57	56	55	54	53	52	51	50	48	47
1	6.4	6.3	6.1	6.0	5.9	5.8	5.7	5.6	5.5	5.4	5.3	5.2	5.1	5.0	4.8	4.7
2	12.8	12.6	12.2	12.0	11.8	11.6	11.4	11.2	11.0	10.8	10.6	10.4	10.2	10.0	9.6	9.4
3	19.2	18.9	18.3	18.0	17.7	17.4	17.1	16.8	16.5	16.2	15.9	15.6	15.3	15.0	14.4	14.1
4	25.6	25.2	24.4	24.0	23.6	23.2	22.8	22.4	22.0	21.6	21.2	20.8	20.4	20.0	19.2	18.8
5	32.0	31.5	30.5	30.0	29.5	29.0	28.5	28.0	27.5	27.0	26.5	26.0	25.5	25.0	24.0	23.5
6	38.4	37.8	36.6	36.0	35.4	34.8	34.2	33.6	33.0	32.4	31.8	31.2	30.6	30.0	28.8	28.2
7	44.8	44.1	42.7	42.0	41.3	40.6	39.9	39.2	38.5	37.8	37.1	36.4	35.7	35.0	33.6	32.9
8	51.2	50.4	48.8	48.0	47.2	46.4	45.6	44.8	44.0	43.2	42.4	41.6	40.8	40.0	38.4	37.6
9	57.6	56.7	54.9	54.0	53.1	52.2	51.3	50.4	49.5	48.6	47.7	46.8	45.9	45.0	43.2	42.3

° ′	log sin	d	log tan	dc	log cot	log cos	d	
15 0	9.4130	47	9.4281	50	0.5719	9.9849	3	0 75
10	9.4177	46	9.4331	50	0.5669	9.9846	3	50
20	9.4223	46	9.4381	49	0.5619	9.9843	4	40
30	9.4269	45	9.4430	49	0.5570	9.9839	3	30
40	9.4314	45	9.4479	48	0.5521	9.9836	4	20
50	9.4359	44	9.4527	48	0.5473	9.9832	4	10
16 0	9.4403	44	9.4575	47	0.5425	9.9828	3	0 74
10	9.4447	44	9.4622	47	0.5378	9.9825	4	50
20	9.4491	42	9.4669	47	0.5331	9.9821	4	40
30	9.4533	43	9.4716	46	0.5284	9.9817	3	30
40	9.4576	42	9.4762	46	0.5238	9.9814	4	20
50	9.4618	41	9.4808	45	0.5192	9.9810	4	10
17 0	9.4659	41	9.4853	45	0.5147	9.9806	4	0 73
10	9.4700	41	9.4898	45	0.5102	9.9802	4	50
20	9.4741	40	9.4943	44	0.5057	9.9798	4	40
30	9.4781	40	9.4987	44	0.5013	9.9794	4	30
40	9.4821	40	9.5031	44	0.4969	9.9790	4	20
50	9.4861	39	9.5075	43	0.4925	9.9786	4	10
18 0	9.4900	39	9.5118	43	0.4882	9.9782	4	0 72
10	9.4939	38	9.5161	42	0.4839	9.9778	4	50
20	9.4977	38	9.5203	42	0.4797	9.9774	4	40
30	9.5015	37	9.5245	42	0.4755	9.9770	5	30
40	9.5052	38	9.5287	42	0.4713	9.9765	4	20
50	9.5090	36	9.5329	41	0.4671	9.9761	4	10
19 0	9.5126	37	9.5370	41	0.4630	9.9757	5	0
10	9.5163	36	9.5411	40	0.4589	9.9752	4	50 71
20	9.5199	36	9.5451	40	0.4549	9.9748	5	40
30	9.5235	35	9.5491	40	0.4509	9.9743	4	30
40	9.5270	36	9.5531	40	0.4469	9.9739	5	20
50	9.5306	35	9.5571	40	0.4429	9.9734	4	10
20 0	9.5341		9.5611		0.4389	9.9730		0 70
	log cos	d	log cot	dc	log tan	log sin	d	′ °

pp

	50	49
1	5.0	4.9
2	10.0	9.8
3	15.0	14.7
4	20.0	19.6
5	25.0	24.5
6	30.0	29.4
7	35.0	34.3
8	40.0	39.2
9	45.0	44.1

	48	47
1	4.8	4.7
2	9.6	9.4
3	14.4	14.1
4	19.2	18.8
5	24.0	23.5
6	28.8	28.2
7	33.6	32.9
8	38.4	37.6
9	43.2	42.3

	46	45
1	4.6	4.5
2	9.2	9.0
3	13.8	13.5
4	18.4	18.0
5	23.0	22.5
6	27.6	27.0
7	32.2	31.5
8	36.8	36.0
9	41.4	40.5

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	40	39	38
1	4.1	4.0	3.9	3.8
2	8.2	8.0	7.8	7.6
3	12.3	12.0	11.7	11.4
4	16.4	16.0	15.6	15.2
5	20.5	20.0	19.5	19.0
6	24.6	24.0	23.4	22.8
7	28.7	28.0	27.3	26.6
8	32.8	32.0	31.2	30.4
9	36.9	36.0	35.1	34.2

	37	36	35
1	3.7	3.6	3.5
2	7.4	7.2	7.0
3	11.1	10.8	10.5
4	14.8	14.4	14.0
5	18.5	18.0	17.5
6	22.2	21.6	21.0
7	25.9	25.2	24.5
8	29.6	28.8	28.0
9	33.3	32.4	31.5

°	′	log sin	d	log tan	dc	log cot	log cos	d	′	°
20	0	9.5341	34	9.5611	39	0.4389	9.9730	5	0	70
	10	9.5375	34	9.5650	39	0.4350	9.9725	4	50	
	20	9.5409	34	9.5689	38	0.4311	9.9721	5	40	
	30	9.5443	34	0.5727	39	0.4273	9.9716	5	30	
	40	9.5477	33	9.5766	38	0.4234	9.9711	5	20	
	50	9.5510	33	9.5804	38	0.4196	9.9706	4	10	
21	0	9.5543	33	9.5842	37	0.4158	9.9702	5	0	69
	10	9.5576	33	9.5879	38	0.4121	9.9697	5	50	
	20	9.5609	32	9.5917	37	0.4083	9.9692	5	40	
	30	9.5641	32	9.5954	37	0.4046	9.9687	5	30	
	40	9.5673	31	9.5991	37	0.4009	9.9682	5	20	
	50	9.5704	32	9.6028	36	0.3972	9.9677	5	10	
22	0	9.5736	31	9.6064	36	0.3936	9.9672	5	0	68
	10	9.5767	31	9.6100	36	0.3900	9.9667	6	50	
	20	9.5798	30	9.6136	36	0.3864	9.9661	5	40	
	30	9.5828	31	9.6172	36	0.3828	9.9656	5	30	
	40	9.5859	30	9.6208	35	0.3792	9.9651	5	20	
	50	9.5889	30	9.6243	36	0.3757	9.9646	6	10	
23	0	9.5919	29	9.6279	35	0.3721	9.9640	5	0	67
	10	9.5948	30	9.6314	34	0.3686	9.9635	6	50	
	20	9.5978	29	9.6348	35	0.3652	9.9629	5	40	
	30	9.6007	29	9.6383	34	0.3617	9.9624	6	30	
	40	9.6036	29	9.6417	35	0.3583	9.9618	5	20	
	50	9.6065	28	9.6452	34	0.3548	9.9613	6	10	
24	0	9.6093	28	9.6486	34	0.3514	9.9607	5	0	66
	10	9.6121	28	9.6520	33	0.3480	9.9602	6	50	
	20	9.6149	28	9.6553	34	0.3447	9.9596	6	40	
	30	9.6177	28	9.6587	33	0.3413	9.9590	6	30	
	40	9.6205	27	9.6620	34	0.3380	9.9584	5	20	
	50	9.6232	27	9.6654	33	0.3346	9.9579	6	10	
25	0	9.6259		9.6687		0.3313	9.9573		0	65
		log cos	d	log cot	dc	log tan	log sin	d	′	°

pp

	4	5	6
1	0.4	0.5	0.6
2	0.8	1.0	1.2
3	1.2	1.5	1.8
4	1.6	2.0	2.4
5	2.0	2.5	3.0
6	2.4	3.0	3.6
7	2.8	3.5	4.2
8	3.2	4.0	4.8
9	3.6	4.5	5.4

	39	38	37	36	35	34	33	32	31	30	29	28	27
1	3.9	3.8	3.7	3.6	3.5	3.4	3.3	3.2	3.1	3.0	2.9	2.8	2.7
2	7.8	7.6	7.4	7.2	7.0	6.8	6.6	6.4	6.2	6.0	5.8	5.6	5.4
3	11.7	11.4	11.1	10.8	10.5	10.2	9.9	9.6	9.3	9.0	8.7	8.4	8.1
4	15.6	15.2	14.8	14.4	14.0	13.6	13.2	12.8	12.4	12.0	11.6	11.2	10.8
5	19.5	19.0	18.5	18.0	17.5	17.0	16.5	16.0	15.5	15.0	14.5	14.0	13.5
6	23.4	22.8	22.2	21.6	21.0	20.4	19.8	19.2	18.6	18.0	17.4	16.8	16.2
7	27.3	26.6	25.9	25.2	24.5	23.8	23.1	22.4	21.7	21.0	20.3	19.6	18.9
8	31.2	30.4	29.6	28.8	28.0	27.2	26.4	25.6	24.8	24.0	23.2	22.4	21.6
9	35.1	34.2	33.3	32.4	31.5	30.6	29.7	28.8	27.9	27.0	26.1	25.2	24.3

° ′	log sin	d	log tan	dc	log cot	log cos	d	′ °
25 0	9.6259	27	9.6687	33	0.3313	9.9573	6	0 65
10	9.6286	27	9.6720	32	0.3280	9.9567	6	50
20	9.6313	27	9.6752	33	0.3248	9.9561	6	40
30	9.6340	26	9.6785	32	0.3215	9.9555	6	30
40	9.6366	26	9.6817	33	0.3183	9.9549	6	20
50	9.6392	26	9.6850	32	0.3150	9.9543	6	10
26 0	9.6418	26	9.6882	32	0.3118	9.9537	7	0 64
10	9.6444	26	9.6914	32	0.3086	9.9530	6	50
20	9.6470	25	9.6946	31	0.3054	9.9524	6	40
30	9.6495	26	9.6977	32	0.3023	9.9518	6	30
40	9.6521	25	9.7009	31	0.2991	9.9512	7	20
50	9.6546	24	9.7040	32	0.2960	9.9505	6	10
27 0	9.6570	25	9.7072	31	0.2928	9.9499	7	0 63
10	9.6595	25	9.7103	31	0.2897	9.9492	6	50
20	9.6620	24	9.7134	31	0.2866	9.9486	7	40
30	9.6644	24	9.7165	31	0.2835	9.9479	6	30
40	9.6668	24	9.7196	30	0.2804	0.9473	7	20
50	9.6692	24	9.7226	31	0.2774	9.9466	7	10
28 0	9.6716	24	9.7257	30	0.2743	9.9459	6	0 62
10	9.6740	23	9.7287	30	0.2713	9.9453	7	50
20	9.6763	24	9.7317	31	0.2683	9.9446	7	40
30	9.6787	23	9.7348	30	0.2652	9.9439	7	30
40	9.6810	23	9.7378	30	0.2622	9.9432	7	20
50	9.6833	23	9.7408	30	0.2592	9.9425	7	10
29 0	9.6856	22	9.7438	29	0.2562	9.9418	7	0 61
10	9.6878	23	9.7467	30	0.2533	9.9411	7	50
20	9.6901	22	9.7497	29	0.2503	9.9404	7	40
30	9.6923	23	9.7526	30	0.2474	9.9397	7	30
40	9.6946	22	9.7556	29	0.2444	9.9390	7	20
50	9.6968	22	9.7585	29	0.2415	9.9383	8	10
30 0	9.6990		9.7614		0.2386	9.9375		0 60
	log cos	d	log cot	dc	log tan	log sin	d	′ °

pp

	6	7	8
1	0.6	0.7	0.8
2	1.2	1.4	1.6
3	1.8	2.1	2.4
4	2.4	2.8	3.2
5	3.0	3.5	4.0
6	3.6	4.2	4.8
7	4.2	4.9	5.6
8	4.8	5.6	6.4
9	5.4	6.3	7.2

	33	32	31	30	29	27	26	25	24	23	22
1	3.3	3.2	3.1	3.0	2.9	2.7	2.6	2.5	2.4	2.3	2.2
2	6.6	6.4	6.2	6.0	5.8	5.4	5.2	5.0	4.8	4.6	4.4
3	9.9	9.6	9.3	9.0	8.7	8.1	7.8	7.5	7.2	6.9	6.6
4	13.2	12.8	12.4	12.0	11.6	10.8	10.4	10.0	9.6	9.2	8.8
5	16.5	16.0	15.5	15.0	14.5	13.5	13.0	12.5	12.0	11.5	11.0
6	19.8	19.2	18.6	18.0	17.4	16.2	15.6	15.0	14.4	13.8	13.2
7	23.1	22.4	21.7	21.0	20.3	18.9	18.2	17.5	16.8	16.1	15.4
8	26.4	25.6	24.8	24.0	23.2	21.6	20.8	20.0	19.2	18.4	17.6
9	29.7	28.8	27.9	27.0	26.1	24.3	23.4	22.5	21.6	20.7	19.8

°	′	log sin	d	log tan	dc	log cot	log cos	d		
30	0	9.6990	22	9.7614	30	0.2386	9.9375	7	0	60
	10	9.7012	21	9.7644	29	0.2356	9.9368	7	50	
	20	9.7033	22	9.7673	28	0.2327	9.9361	8	40	
	30	9.7055	21	9.7701	29	0.2299	9.9353	7	30	
	40	9.7076	21	9.7730	29	0.2270	9.9346	8	20	
	50	9.7097	21	9.7759	29	0.2241	9.9338	7	10	
31	0	9.7118	21	9.7788	28	0.2212	9.9331	8	0	59
	10	9.7139	21	9.7816	29	0.2184	9.9323	8	50	
	20	9.7160	21	9.7845	28	0.2155	9.9315	7	40	
	30	9.7181	20	9.7873	29	0.2127	9.9308	8	30	
	40	9.7201	21	9.7902	28	0.2098	9.9300	8	20	
	50	9.7222	20	9.7930	28	0.2070	9.9292	8	10	
32	0	9.7242	20	9.7958	28	0.2042	9.9284	8	0	58
	10	9.7262	20	9.7986	28	0.2014	9.9276	8	50	
	20	9.7282	20	9.8014	28	0.1986	9.9268	8	40	
	30	9.7302	20	9.8042	28	0.1958	9.9260	8	30	
	40	9.7322	20	9.8070	27	0.1930	9.9252	8	20	
	50	9.7342	19	9.8097	28	0.1903	9.9244	8	10	
33	0	9.7361	19	9.8125	28	0.1875	9.9236	8	0	57
	10	9.7380	20	9.8153	27	0.1847	9.9228	9	50	
	20	9.7400	19	9.8180	28	0.1820	9.9219	8	40	
	30	9.7419	19	9.8208	27	0.1792	9.9211	8	30	
	40	9.7438	19	9.8235	28	0.1765	9.9203	9	20	
	50	9.7457	19	9.8263	27	0.1737	9.9194	8	10	
34	0	9.7476	18	9.8290	27	0.1710	9.9186	9	0	56
	10	9.7494	19	9.8317	27	0.1683	9.9177	8	50	
	20	9.7513	18	9.8344	27	0.1656	9.9169	9	40	
	30	9.7531	19	9.8371	27	0.1629	9.9160	9	30	
	40	9.7550	18	9.8398	27	0.1602	9.9151	9	20	
	50	9.7568		9.8425		0.1575	9.9142		10	
35	0	9.7586	18	9.8452	27	0.1548	9.9134	8	0	55
		log cos	d	log cot	dc	log tan	log sin	d	′	°

pp

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	30	29	28
1	3.0	2.9	2.8
2	6.0	5.8	5.6
3	9.0	8.7	8.4
4	12.0	11.6	11.2
5	15.0	14.5	14.0
6	18.0	17.4	16.8
7	21.0	20.3	19.6
8	24.0	23.2	22.4
9	27.0	26.1	25.2

	27	22	21
1	2.7	2.2	2.1
2	5.4	4.4	4.2
3	8.1	6.6	6.3
4	10.8	8.8	8.4
5	13.5	11.0	10.5
6	16.2	13.2	12.6
7	18.9	15.4	14.7
8	21.6	17.6	16.8
9	24.3	19.8	18.9

	20	19	18
1	2.0	1.9	1.8
2	4.0	3.8	3.6
3	6.0	5.7	5.4
4	8.0	7.6	7.2
5	10.0	9.5	9.0
6	12.0	11.4	10.8
7	14.0	13.3	12.6
8	16.0	15.2	14.4
9	18.0	17.1	16.2

°	′	log sin	d	log tan	dc	log cot	log cos	d	′	°
35	0	9.7586	18	9.8452	27	0.1548	9.9134	9	0	55
	10	9.7604	18	9.8479	27	0.1521	9.9125	9	50	
	20	9.7622	18	9.8506	27	0.1494	9.9116	9	40	
	30	9.7640	17	9.8533	26	0.1467	9.9107	9	30	
	40	9.7657	18	9.8559	27	0.1441	9.9098	9	20	
	50	9.7675	17	9.8586	27	0.1414	9.9089	9	10	
36	0	9.7692	18	9.8613	26	0.1387	9.9080	10	0	54
	10	9.7710	17	9.8639	27	0.1361	9.9070	9	50	
	20	9.7727	17	9.8666	26	0.1334	9.9061	9	40	
	30	9.7744	17	9.8692	26	0.1308	9.9052	10	30	
	40	9.7761	17	9.8718	27	0.1282	9.9042	9	20	
	50	9.7778	17	9.8745	26	0.1255	9.9033	10	10	
37	0	9.7795	16	9.8771	26	0.1229	9.9023	9	0	53
	10	9.7811	17	9.8797	27	0.1203	9.9014	10	50	
	20	9.7828	16	9.8824	26	0.1176	9.9004	9	40	
	30	9.7844	17	9.8850	26	0.1150	9.8995	10	30	
	40	9.7861	16	9.8876	26	0.1124	9.8985	10	20	
	50	9.7877	16	9.8902	26	0.1098	9.8975	10	10	
38	0	9.7893	17	9.8928	26	0.1072	9.8965	10	0	52
	10	9.7910	16	9.8954	26	0.1046	9.8955	10	50	
	20	9.7926	15	9.8980	26	0.1020	9.8945	10	40	
	30	9.7941	16	9.9006	26	0.0994	9.8935	10	30	
	40	9.7957	16	9.9032	26	0.0968	9.8925	10	20	
	50	9.7973	16	9.9058	26	0.0942	9.8915	10	10	
39	0	9.7989	15	9.9084	26	0.0916	9.8905	10	0	51
	10	9.8004	16	9.9110	25	0.0890	9.8895	11	50	
	20	9.8020	15	9.9135	26	0.0865	9.8884	10	40	
	30	9.8035	15	9.9161	26	0.0839	9.8874	10	30	
	40	9.8050	16	9.9187	25	0.0813	9.8864	11	20	
	50	9.8066	15	9.9212	26	0.0788	9.8853	10	10	
40	0	9.8081		9.9238		0.0762	9.8843		0	50
		log cos	d	log cot	dc	log tan	log sin	d	′	°

pp

	9	10	11	27	26	25	18	17	16	15
1	0.9	1.0	1.1	2.7	2.6	2.5	1.8	1.7	1.6	1.5
2	1.8	2.0	2.2	5.4	5.2	5.0	3.6	3.4	3.2	3.0
3	2.7	3.0	3.3	8.1	7.8	7.5	5.4	5.1	4.8	4.5
4	3.6	4.0	4.4	10.8	10.4	10.0	7.2	6.8	6.4	6.0
5	4.5	5.0	5.5	13.5	13.0	12.5	9.0	8.5	8.0	7.5
6	5.4	6.0	6.6	16.2	15.6	15.0	10.8	10.2	9.6	9.0
7	6.3	7.0	7.7	18.9	18.2	17.5	12.6	11.9	11.2	10.5
8	7.2	8.0	8.8	21.6	20.8	20.0	14.4	13.6	12.8	12.0
9	8.1	9.0	9.9	24.3	23.4	22.5	16.2	15.3	14.4	13.5

°	′	log sin	d	log tan	dc	log cot	log cos	d	′	°
40	0	9.8081	15	9.9238	26	0.0762	9.8843	11	0	50
	10	9.8096	15	9.9264	25	0.0736	9.8832	11	50	
	20	9.8111	14	9.9289	26	0.0711	9.8821	11	40	
	30	9.8125	15	9.9315	26	0.0685	9.8810	10	30	
	40	9.8140	15	9.9341	25	0.0659	9.8800	11	20	
	50	9.8155	14	9.9366	26	0.0634	9.8789	11	10	
41	0	9.8169	15	9.9392	25	0.0608	9.8778	11	0	49
	10	9.8184	14	9.9417	26	0.0583	9.8767	11	50	
	20	9.8198	15	9.9443	25	0.0557	9.8756	11	40	
	30	9.8213	14	9.9468	26	0.0532	9.8745	12	30	
	40	9.8227	14	9.9494	25	0.0506	9.8733	11	20	
	50	9.8241	14	9.9519	25	0.0481	9.8722	11	10	
42	0	9.8255	14	9.9544	26	0.0456	9.8711	12	0	48
	10	9.8269	14	9.9570	25	0.0430	9.8699	11	50	
	20	9.8283	14	9.9595	26	0.0405	9.8688	12	40	
	30	9.8297	14	9.9621	25	0.0379	9.8676	11	30	
	40	9.8311	13	9.9646	25	0.0354	9.8665	12	20	
	50	9.8324	14	9.9671	26	0.0329	9.8653	12	10	
43	0	9.8338	13	9.9697	25	0.0303	9.8641	12	0	47
	10	9.8351	14	9.9722	25	0.0278	9.8629	11	50	
	20	9.8365	13	9.9747	25	0.0253	0.8618	12	40	
	30	9.8378	13	9.9772	26	0.0228	9.8606	12	30	
	40	9.8391	14	9.9798	25	0.0202	9.8594	12	20	
	50	9.8405	13	9.9823	25	0.0177	9.8582	13	10	
44	0	9.8418	13	9.9848	26	0.0152	9.8569	12	0	46
	10	9.8431	13	9.9874	25	0.0126	9.8557	12	50	
	20	9.8444	13	9.9899	25	0.0101	9.8545	13	40	
	30	9.8457	12	9.9924	25	0.0076	0.8532	12	30	
	40	9.8469	13	9.9949	26	0.0051	9.8520	13	20	
	50	9.8482	13	9.9975	25	0.0025	9.8507	12	10	
45	0	9.8495		0.0000		0.0000	9.8495		0	45
		log cos	d	log cot	dc	log tan	log sin	d	′	°

pp

	10	11	12
1	1.0	1.1	1.2
2	2.0	2.2	2.4
3	3.0	3.3	3.6
4	4.0	4.4	4.8
5	5.0	5.5	6.0
6	6.0	6.6	7.2
7	7.0	7.7	8.4
8	8.0	8.8	9.6
9	9.0	9.9	10.8

	26	25	15	14	13
1	2.6	2.5	1.5	1.4	1.3
2	5.2	5.0	3.0	2.8	2.6
3	7.8	7.5	4.5	4.2	3.9
4	10.4	10.0	6.0	5.6	5.2
5	13.0	12.5	7.5	7.0	6.5
6	15.6	15.0	9.0	8.4	7.8
7	18.2	17.5	10.5	9.8	9.1
8	20.8	20.0	12.0	11.2	10.4
9	23.4	22.5	13.5	12.6	11.7

Deg.	Radians	n sin	n csc	n tan	n cot	n sec	n cos		
0	0.0000	.000	......	.000	......	1.000	1.00	1.5708	90
1	0.0175	.017	57.3	.017	57.3	1.000	1.00	1.5533	89
2	0.0349	.035	28.7	.035	28.6	1.001	.999	1.5359	88
3	0.0524	.052	19.1	.052	19.1	1.001	.999	1.5184	87
4	0.0698	.070	14.3	.070	14.3	1.002	.998	1.5010	86
5	0.0873	.087	11.5	.087	11.4	1.004	.996	1.4835	85
6	0.1047	.105	9.57	.105	9.51	1.006	.995	1.4661	84
7	0.1222	.122	8.21	.123	8.14	1.008	.993	1.4486	83
8	0.1396	.139	7.19	.141	7.12	1.010	.990	1.4312	82
9	0.1571	.156	6.39	.158	6.31	1.012	.988	1.4137	81
10	0.1745	.174	5.76	.176	5.67	1.015	.985	1.3963	80
11	0.1920	.191	5.24	.194	5.14	1.019	.982	1.3788	79
12	0.2094	.208	4.81	.213	4.70	1.022	.978	1.3614	78
13	0.2269	.225	4.45	.231	4.33	1.026	.974	1.3439	77
14	0.2443	.242	4.13	.249	4.01	1.031	.970	1.3265	76
15	0.2618	.259	3.86	.268	3.73	1.035	.966	1.3090	75
16	0.2793	.276	3.63	.287	3.49	1.040	.961	1.2915	74
17	0.2967	.292	3.42	.306	3.27	1.046	.956	1.2741	73
18	0.3142	.309	3.24	.325	3.08	1.051	.951	1.2566	72
19	0.3316	.326	3.07	.344	2.90	1.058	.946	1.2392	71
20	0.3491	.342	2.92	.364	2.75	1.064	.940	1.2217	70
21	0.3665	.358	2.79	.384	2.61	1.071	.934	1.2043	69
22	0.3840	.375	2.67	.404	2.48	1.079	.927	1.1868	68
23	0.4014	.391	2.56	.424	2.36	1.086	.921	1.1694	67
24	0.4189	.407	2.46	.445	2.25	1.095	.914	1.1519	66
25	0.4363	.423	2.37	.466	2.14	1.103	.906	1.1345	65
26	0.4538	.438	2.28	.488	2.05	1.113	.899	1.1170	64
27	0.4712	.454	2.20	.510	1.96	1.122	.891	1.0996	63
28	0.4887	.469	2.13	.532	1.88	1.133	.883	1.0821	62
29	0.5061	.485	2.06	.554	1.80	1.143	.875	1.0647	61
30	0.5236	.500	2.00	.577	1.73	1.155	.866	1.0472	60
31	0.5411	.515	1.94	.601	1.66	1.167	.857	1.0297	59
32	0.5585	.530	1.89	.625	1.60	1.179	.848	1.0123	58
33	0.5760	.545	1.84	.649	1.54	1.192	.839	0.9948	57
34	0.5934	.559	1.79	.675	1.48	1.206	.829	0.9774	56
35	0.6109	.574	1.74	.700	1.43	1.221	.819	0.9599	55
36	0.6283	.588	1.70	.727	1.38	1.236	.809	0.9425	54
37	0.6458	.602	1.66	.754	1.33	1.252	.799	0.9250	53
38	0.6632	.616	1.62	.781	1.28	1.269	.788	0.9076	52
39	0.6807	.629	1.59	.810	1.23	1.287	.777	0.8901	51
40	0.6981	.643	1.56	.839	1.19	1.305	.766	0.8727	50
41	0.7156	.656	1.52	.869	1.15	1.325	.755	0.8552	49
42	0.7330	.669	1.49	.900	1.11	1.346	.743	0.8378	48
43	0.7505	.682	1.47	.933	1.07	1.367	.731	0.8203	47
44	0.7679	.695	1.44	.966	1.04	1.390	.719	0.8029	46
45	0.7854	.707	1.41	1.00	1.00	1.414	.707	0.7854	45
		n cos	n sec	n cot	n tan	n csc	n sin	Radians	Deg.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

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